Analytical Methods for Conformal Field Theory

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I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject of Physics elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 126/2014 on 18/11/2014.

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Zusammenfassung

In dieser Dissertation stellen wir moderne analytische Methoden zur Untersuchung von konformen Feldtheorien (CFTs) in mehr als zwei Dimensionen vor. Mit Hilfe dieser Methoden können das Spektrum der Theorie und die Operatorprodukt-Koeffizienten (OPE-Koeffizienten) ermittelt werden. Zunächst untersuchen wir das Spektrum lokaler Operatoren in CFTs auf einem Defekt mit Kodimension größer eins. Wir zeigen, dass für großen transversalen Spin s das Spektrum jeder Theorie abzählbar unendlich viele Häufungspunkte aufweist. Der Spin s ist die Quantenzahl, die zu der Untergruppe der Lorentzgruppe gehört, welche den Defekt invariant lässt. Des Weiteren finden wir die OPE-Koeffizienten und die anomalen Dimensionen der zu den Häufungspunkten gehörenden Operatoren in einer Entwicklung in $\frac{1}{s}$ mit Hilfe von Lichtkegel-Bootstrap-Techniken. Außerdem leiten wir aus der Diskontinuität der kausalen Zweipunktfunktion die Operatordimensionen und OPE-Koeffizienten als analytische Funktionen von s her. Im zweiten Teil dieser Arbeit führen wir die Mellindarstellung von konformen Korrelationsfunktionen ein. In dieser Darstellung sind das Spektrum und die OPE-Koeffizienten manifest enthalten. Wir legen den Fokus auf die Beschreibung von Vierpunktfunktionen in drei Dimensionen von entweder ausschließlich Spin $\frac{1}{2}$ Operatoren oder einer Mischung aus Spin $\frac{1}{2}$ und skalaren Operatoren. Nachdem wir für diese Vierpunktfunktionen die Mellinamplituden definieren, untersuchen wir die Polstruktur dieser genauer. Im Anschluss illustrieren wir die Analyse an konkreten Mellinamplituden von fermionischen Wittendiagrammen und konformen fermionischen Feynmandiagrammen. Im letzten Teil untersuchen wir die OPE im Kontext der Holographie. Hierbei leiten wir theorieunabhängige Beziehungen zwischen den OPE-Koeffizienten der Weltflächen-CFT einer Stringtheorie in Anti-de-Sitter-Raumzeit und der dualen CFT her.

Abstract

In this thesis, we discuss some modern analytical approaches to studying conformal field theories (CFTs) in dimensions greater than two. The results thus derived pertain to the dynamical data that define a generic CFT, namely the spectrum of operators and the coefficients in the operator product expansion (OPE). We begin with an investigation of the spectrum of local operators supported on conformal defects of codimension greater than one and establish the existence therein of a countably infinite number of universal accumulation points at large transverse spin s. Here, s is a quantum number associated with the symmetry under the Lorentz transformations that preserve the defect. Using lightcone bootstrap techniques, we calculate the anomalous dimensions and OPE coefficients of the operators that populate these accumulation points in a large s expansion. Furthermore, we derive an integral formula to obtain the CFT data associated with the defect theory from the discontinuity in the causal two-point function of scalar operators in the ambient theory, thereby inverting the expansion of this correlator in the defect channel. This formula extracts the operator dimensions and OPE coefficients in an analytic function in s and also enables us to resum the large s expansion obtained using lightcone bootstrap. Thereafter we move on to a discussion of the Mellin representation of fermionic conformal correlators. The dynamical data in CFTs is manifest in the analytic properties of Mellin amplitudes. We define, concretely for three spacetime dimensions, the Mellin amplitudes associated with the four-point function of spin-half operators and the mixed four-point function of spin-half and scalar operators. We analyze the pole structure of these Mellin amplitudes and illustrate the general features thus unraveled with some explicit computations of Mellin amplitudes associated with Witten diagrams and conformal Feynman integrals with fermionic legs. Finally we look at the OPE in the context of holography and derive a set of theory independent relations between OPE coefficients in the worldsheet CFT of a string theory in anti-de Sitter spacetime and those in the dual CFT.

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Publications by the author

This thesis is based on the following publications by the author:

[1] S. Ghosh, S. Sarkar, and M. Verma, "Implications of the AdS/CFT correspondence on Spacetime and Worldsheet OPE coefficients," arXiv:1703.06132 [hep-th].

[2] J. Faller, S. Sarkar, and M. Verma, "Mellin Amplitudes for Fermionic Conformal Correlators," *JHEP* **03** (2018) 106, arXiv:1711.07929 [hep-th].

[3] M. Lemos, P. Liendo, M. Meineri, and S. Sarkar, "Universality at large transverse spin in defect CFT," arXiv:1712.08185[hep-th].

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Chapter 1

Introduction

Quantum field theory (QFT) is an important framework in physics that has been extremely useful in condensed matter physics and forms the very basis of modern particle physics. The first successful physical model based on quantum field theory was quantum electrodynamics (QED) that governs the dynamics of the eletromagnetic field and particles interacting with it via a charge. QED remains one of the most precisely tested physical theories so far. This framework was improved upon when gauge symmetry in QFT was extended to non-abelian groups to describe strong and weak nuclear forces thus giving us the Standard Model of particle physics. A consistent quantum mechanical description of gravitational interactions still remains elusive. However, the efforts directed towards discovering a quantum theory of gravity have led us to string theory and the AdS/CFT correspondence. The framework of QFT has also been successfully employed in studying statistical systems at criticality, thus underlining the general appeal of QFT to systems with infinitely many degrees of freedom.

A quantum field theory describing a physical system should be interpreted as an effective theory describing the dynamics in the degrees of freedom that can be detected at a given length scale (or equivalently an energy scale). For example in particle physics, a particular matter particle can be studied only at a certain energy scale. The imposition of a momentum cut-off is equivalent to approximating spacetime with a lattice. The QFT describing the dynamics of this matter particle is associated with the length scale defined by this lattice. To calculate observables at longer length scales, we would have to integrate out the high momentum degrees of freedom as required which corresponds to coarse graining over the fields. This evolution of the parameters describing the theory with change in the characteristic scale of the theory is described by the renormalization group (RG) equations. As a theory flows down to large distance or low energy scales, known as the infrared (IR) limit, the effective description either breaks down as the parameters keep growing, or the theory flows to a fixed point where changing the characteristic scale leaves the theory invariant. The fixed point theory in the IR may be a trivial theory of free massless particles or an interacting theory with a continuous spectrum.

Let us now consider the other limit of the RG flow corresponding to high energy or equivalently infinitesimal length scales known as the ultraviolet (UV) limit. A generic QFT at intermediate length scales may have a physical cut-off at short distance scales and thus it may not be described by a QFT in the UV. This is referred to as the theory not having a UV completion. In high energy physics, we ideally want our theory to be defined at all energy scales and thus have a consistent UV completion. The UV completion is also a fixed point which can be deformed by a relevant operator to flow down to the effective theory in consideration.

Fixed point theories turn out to be even more interesting as scale invariance in QFT is usually assumed to be enhanced to conformal symmetry thus making a fixed point theory a conformal field theory (CFT). This assumption for local theories is motivated by the fact that conformal transformations are coordinate transformations for which the associated Jacobian matrix is proportional to a rotation matrix (in Euclidean spacetime) and thus every conformal transformation locally resembles a composition of rotations and a rescaling. This statement has also been proved in two and four dimensions for unitary theories [4–7] but a complete elucidation of this phenomenon in general dimensions is still lacking.

It can thus be expected that all UV complete QFTs lie on RG trajectories with CFTs at the end points. This offers us a fresh perspective to the study of QFT and highlights the importance of CFTs as studying the space of QFTs corresponds to mapping out the space of all CFTs. The presence of conformal symmetry in fixed point theories makes them more tractable compared to generic Poincare invariant QFTs. This is even more so in two dimensions, where the conformal algebra is the infinite dimensional Virasoro algebra. We shall however be discussing CFTs in three or higher number of dimensions in this thesis.

An important phenomenon emerging from RG flows to an IR fixed point is critical universality. As we zoom out to longer length scales and tend to the IR CFT, we lose information on the effective theories (or the UV completion) at shorter length scales. Consequently, many different UV theories (or effective theories on different RG trajectories) may flow down to a common IR fixed point. This IR equivalence of different theories is the phenomenon of critical universality. For example, the Wilson-Fisher fixed point, the critical Ising model in three dimensions and the theory governing the liquid-vapor phase transition of water are all the same CFT. The relevance of a particular CFT to a number of different microscopic realizations via the phenomenon of universality provides us with further motivation to study CFTs.

The physicists' dream for a quantum theory of gravity that is well-defined at all energy scales has given us yet another reason to pursue the exploration of CFT. This is the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence [8–10], which in its general form states that string theory in $d+1$ dimensional anti-de Sitter (AdS) spacetime is exactly dual to a CFT living in d dimensional flat spacetime. Although this statement has not yet been proved, it has tremendously influenced research in theoretical high energy physics. It allows us insights into the nature of quantum gravity through the dual CFT and vice-versa it enables us to study dynamics in strongly coupled CFTs through the more tractable semi-classical limit of the dual theory in AdS.

We can thus be convinced that the study of CFT is of crucial importance to the study of QFT and its different realizations in particle physics, statistical systems and even quantum gravity. The phenomenon of critical universality encourages us to attach greater importance to a particular CFT defined non-perturbatively than to any of its different possible realizations. It is possible to define a generic CFT in this manner owing to the fact that the operator product expansion (OPE) in CFT has a finite radius of convergence [11–16]. Consequently, all correlation functions in a CFT can be expanded in terms of two-point functions, the only dynamical data going into this expansion being the operator dimensions Δ that determine the two-point functions completely and the coefficients in the OPE. This approach to CFT entails defining the theory with the data on the spectrum of operator dimensions and the corresponding OPE coefficients (collectively referred to as CFT data) rather than referring to a Lagrangian description. This is the rationale behind the immensely successful bootstrap program in CFT [17–33] where one exploits the generic properties of the theory like locality, unitarity, the consequences of the conformal symmetry and global symmetries, and the associated consistency conditions, to study the dynamics of the theory.

This thesis too derives inspiration from this philosophy and is thus thematically centered around CFT data. We shall study certain aspects of the consistency conditions in CFTs and their implications, representations of conformal correlators that make CFT data manifest and some properties satisfied by this data in a holographic setting. More specifically, we shall discuss how consistency conditions can be used to calculate operator dimensions and OPE coefficients in CFTs with defects. On the way, we shall unravel some universal features of the spectrum of the defect theories and establish interesting mathematical properties of the associated CFT data. We shall then engage in a discussion of the Mellin representation of fermionic conformal correlators. The Mellin representation makes the CFT data encoded in conformal correlators manifest in the analytic structure of the associated Mellin amplitudes and is of singular importance to conformal gauge theories that admit a $\frac{1}{N}$ expansion, N being the number of colors. Finally, we shall discuss some model independent relations satisfied by OPE coefficients in d dimensional CFTs and OPE coefficients in the worldsheet CFT of the dual string theory in AdS_{d+1}. Let us now delve a bit deeper into each one of these topics separately.

Defects in conformal field theories

Although a conventional discussion of QFT would mostly focus on local operators and their correlation functions, it is also interesting from both theoretical and experimental vantage points to study non-local operators supported on a submanifold in spacetime, otherwise known as defects. It may or may not be possible to represent these non-local operators using the fundamental operators of the ambient theory - see [34] for a review of such constructions. Typical examples of defects are boundary conditions on operators, Wilson and 't Hooft operators in gauge theories [35, 36] and D-branes in string theory. Defects serve as probes to study the dynamics of a theory as all correlation functions are now measured in the presence of this defect. In Lagrangian theories it amounts to an evaluation of all path integrals with an extra insertion corresponding to the defect. The importance of incorporating defects is perhaps even more evident from an experimental point of view as all physical systems may have impurities and are also finite and thus restricted by boundaries.

Defects in CFTs that preserve a part of the conformal symmetry are referred to as conformal defects. In this thesis, we shall be interested in flat (or conformally flat) conformal defects and the associated residual symmetry group consists of the conformal transformations parallel to the defect and Lorentz transformations that fix the defect. A conformal defect also admits local operators that live on the hyperplane that supports the defect. These operators are simply like operators in a lower dimensional CFT carrying some global symmetry quantum numbers. The spectrum of local operators on the defect (or defect operators in short) is closed under the OPE and thus constitute a CFT living on the defect. Although this "defect CFT" satisfies crossing symmetry and unitarity, it differs from the ambient CFT in that it does not have a conserved stress tensor of its own which is the hallmark of locality in a theory. This is because the defect theory is interacting with the ambient CFT and defect operators (and the defect itself) have non-zero correlation functions with operators in the ambient theory. We can think about CFTs admitting conformal defects from the point of view of RG flows in the following manner. If we couple a QFT to another living on a hyperplane and flow towards the IR, the information on the modification in the theory is either lost along the RG flow or we reach the critical point of the ambient theory[1] now with a conformal defect. A test of this hypothesis in the context of the twist line defect in the 3D Ising model is presented in [37] (see also [38]). Note that generically one only expects a scale invariant defect which may or may not be enhanced to a conformal defect. The relation between scale and conformal symmetry for defects is relatively less explored - see [39] for efforts in this direction.

The most well studied conformal defects are boundaries and interface CFTs [40–48]. In the context of holography, conformal defects have been constructed using D-brane systems in AdS in [49, 50, 50] and the dual defect CFT has been directly studied in [51–54]. Correlation functions in the 3D Ising CFT restricted by a spherical boundary were looked at in [55] while the twist line defect was studied in [37, 56]. Correlation functions of defects (non-local operators) with ambient space operators and its operator product expansion have been studied in the context of Wilson and 't Hooft operators in [35, 57–60] and in an abstract setting in [61, 62]. Kinematics of CFTs with conformal defects have been studied in [63–65] for application to correlators of local operators in the ambient and defect theories.

In recent times, we have seen a significant amount of progress made in constraining and solving CFTs based on universal properties of local, unitary CFTs such as crossing symmetry of correlation functions, the existence of a conserved stress tensor and conserved currents (corresponding to global symmetries) in the spectrum owing to locality and universal lower bounds on dimensions of operators [66–70] coming from unitarity concerns. The program of the numerical conformal bootstrap [18–26] has implemented crossing symmetry in a Euclidean configuration aided by the other generic features to make precise predictions of low-lying operator dimensions. Inspired by this success, the conformal bootstrap program has also been extended to the context of defect CFT to constrain the data associated with the ambient theory, the defect theory and their interactions [3, 48, 56, 63, 71–73].

In a generic strongly interacting theory that does not admit a perturbative expansion in a small parameter, it would typically not be possible to obtain analytical results unless the theory is heavily constrained by symmetries. However, crossing symmetry of the four-point function with a pair of almost lightlike separated operators was shown to imply that every CFT admits a large spin expansion [27, 28]. It was shown that the spectrum of every CFT

[1] We shall refer to the ambient CFT and local operators therein also as the "bulk theory" and "bulk operators" respectively. The usage of the word "bulk" here differs from its usage in the context of holography.

features the so-called "double twist operators" which are composite operators with vanishing anomalous dimensions for large values of spin l. The large spin asymptotic expansion served to calculate the anomalous dimensions and OPE coefficients of these composite operators. This development inspired a plethora of work dedicated to understand and apply the large spin expansion in CFT [29–33]. These results were put on a mathematically firm footing when Caron-Huot derived a Lorentzian inversion formula for the OPE and proved that CFT data can be obtained from an analytic function in spin for values of spin generically as low as two [74] (see also [3, 75, 76]). This immediately established that the large spin expansion discussed earlier was not just asymptotic but convergent and indeed the inversion formula allowed us to resum the infinite expansions obtained earlier from lightcone bootstrap. Combining the numerical and analytical approaches to conformal bootstrap can be a powerful tool to obtain information on CFT data as shown by Simons-Duffin in the context of the Ising CFT in three dimensions [33]. In chap. 3 of this thesis, we discuss analogous analytical approaches and results thus obtained for defect CFTs.

Mellin representation of fermionic correlators

So far we have discussed some important advances at understanding the dynamics in CFTs using methods based on the position space representation of conformal correlators. One may still wonder if position space coordinates (or invariants built out of them) are the best choice of variables to work with. We are well aware of the virtues of Fourier transforming the position space correlators in massive QFT to the momentum space representation. Important properties of physically meaningful QFTs like locality, causality and unitarity can be understood from the analytic properties of momentum space representation of correlation functions. For example, from the Källén-Lehmann spectral representation, we know that the two-point function in momentum space has poles corresponding to single particle states and cuts corresponding to multi-particle states in the spectrum of the theory. In general, momentum space amplitudes factorize on poles to lower point amplitudes, a property that manifests itself in the famed BCFW recursion relations [77, 78]. Furthermore, Feynman rules simplify dramatically in momentum space as tree level amplitudes are just a product of momentum space propagators while loop level diagrams involve integrals over the loop momentum with the integrand again given by a product of propagators. This naturally leads us to the question if resorting to momentum space also offers special benefits in studying CFTs as well. An interacting CFT has a continuous mass spectrum. In the spectral representation of the two-point function, this manifests itself as a branch cut extending up to the origin. Furthermore, the action of the conformal group in momentum space is non-linear. Thus the momentum space representation is not preferable for CFT.

In the radial quantization of CFT, the dilatation operator is the Hamiltonian and in unitary theories it can always be diagonalized with the eigenvalues being the scaling dimensions. In dimensions higher than two, this spectrum of eigenvalues is discrete. One can guess that a representation of the conformal correlation functions that makes this discrete spectrum manifest in a spectral representation will be the natural analogue of the momentum space representation in massive QFT and may offer similar advantages in studying CFT. It was shown by Mack [79, 80] that the Mellin representation of conformal correlators makes this discrete spectrum manifest as operator dimensions and OPE coefficients are encoded in the poles and residues (respectively) of Mellin amplitudes. Mellin amplitudes factorize on their poles onto lower point Mellin amplitudes much like momentum space scattering amplitudes. In the context of conformal large N gauge theories, Mellin amplitudes are particularly important as they are meromorphic functions of the Mellin variables that encode information on only the single trace operators in the spectrum.

Following Mack, the Mellin representation of conformal correlators was further developed and its importance to large N gauge theories was highlighted in [81–83]. The flat space limit of Mellin amplitudes was discovered in [81, 84, 85] that concretely relates them to scattering amplitudes in massive QFT. The computational power of the Mellin representation was demonstrated in the context of tree level Witten diagrams in [86, 87] along with some progress with loop level Witten diagrams in [88–91]. More recently, exact holographic correlators have been derived using the Mellin formalism in [92–96]. Mellin amplitudes for conformal Feynman integrals were calculated in [97, 98] and Feynman rules for tree level diagrams with scalar legs were derived in [99]. In the context of higher-spin holography, there have been attempts at examining the non-locality in the bulk interactions using the Mellin representation for the correlators in the dual free CFT [100–102]. The flat space limit of Mellin amplitudes has been used to relate conformal bootstrap to S-matrix bootstrap in [103]. Furthermore, a new approach to the conformal bootstrap has been developed in Mellin space [104, 105] and successfully employed to a number of problems [106–109]. The Mellin representation was extended to incorporate defects in [110, 111]. Recently, there have also been attempts to obtain a Mellin space version of OPE inversion formula [112]. The Mellin representation for spinning conformal correlators is a territory rather less explored. Mellin amplitudes for correlators of scalars with one integer spin operator were defined in [85] (see also [113]). Chap. 4 of this thesis presents our work on Mellin amplitudes for conformal correlators involving fermionic operators. The fermionic sector of a CFT is not captured in the OPE of scalars and hence it is necessary to work with Mellin amplitudes for correlators of spin-half fermions if we have to access the data for this sector using the analytic properties of Mellin amplitudes.

The AdS/CFT correspondence

Black holes serve as perhaps the most important theoretical laboratories for physicists to study gravity. Black holes as predicted by general theory of relativity feature a tantalizing singularity in the fabric of spacetime. Furthermore, black holes behave as thermodynamic objects that radiate energy at non-zero temperatures and have an entropy proportional to the area of the event horizon. The breakdown of classical physics in the interior of a black hole and the emergence of a thermodynamic behavior indicate at the existence of a quantum statistical theory to explain the physics in the interior of a black hole. The area law prescription for black hole entropy further suggests that the spacetime based gravitational theory should emerge from a reorganization of the degrees of freedom in a theory living in one lower dimension. This is the principle of holography.

The AdS/CFT correspondence [8–10] is an explicit realization of the ideas of holography. It states that there is an exact equality between the partition function of a string theory in AdS spacetime and that of a CFT living on its conformally flat boundary. This should allow us to study scattering in the gravitational theory in AdS_{d+1} at arbitrarily high energy scales from the dynamics of the non-gravitational dual CFT_d. In particular, it should allow us to explain the entropy of a black hole with a microscopic theory.

The AdS/CFT correspondence has been subject to numerous tests since its inception but a rigorous proof of the conjecture is still lacking. The AdS/CFT correspondence been tested and studied most often in the regime where the AdS spacetime is weakly curved and one is justified in approximating the AdS theory with its classical limit. Exploration of this conjectured duality beyond this limit has been primarily restricted to the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ [114–129] in which case both the boundary CFT and the worldsheet CFT of the string dual in AdS are under control. There have been some interesting efforts at studying the worldsheet duals of free gauge theories on the boundary [130–135] in general dimensions. Except for the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ however, the worldsheet theory is not as tractable thus complicating the study of the duality in the stringy regime. It is therefore interesting to investigate if we can reproduce features of the AdS string theory from the dual CFT and vice-versa based on general theory independent considerations alone.

One of the most important features of a CFT is the existence of an OPE with a finite radius of convergence. There has been detailed work on the calculation and OPE analysis of correlation functions in the boundary CFT calculated through bulk supergravity [136–144]. It was shown in [145] that the OPE in the boundary CFT can also be explained in terms of physics in the worldsheet CFT of the dual string theory in AdS. In particular, they obtained the contribution of single trace operators to the OPE of scalars in the boundary CFT from the OPE of the dual vertex operators in the worldsheet theory. In chap. 5 of this thesis, we discuss a generalization of their analysis to spinning operators and derive a set of relations between OPE coefficients in the boundary CFT and those in the dual worldsheet theory.

1.1 Summary and outline of the thesis

In chap. 2 of this thesis, we review some of the basic concepts of CFTs in dimensions higher than two and discuss some modern tools and techniques to study them. We also introduce the physics of conformal defects in further details and discuss how the key ideas of CFT generalize to incorporate defects. This includes a discussion of conformal transformations (sec. 2.1), operators and states in a CFT (sec. 2.2), conformal correlators and the embeddding space formalism (sec. 2.3), operator product expansion (sec. 5), conformal blocks (sec. 2.5) and crossing symmetry and other generic properties of a CFT (sec. 2.6), each topic being discussed in the context of CFTs with and without defects. The brief review of the embedding formalism for spinors in three dimensions in sec. 2.3.4 is excerpted from the author's review of the topic in [2].

In chap. 3, we consider CFTs with a defect of codimension greater than one and show that every such defect theory admits a large "transverse spin" s expansion. Transverse spin is the quantum number associated with the Lorentz transformations in the bulk that preserve the defect. Concretely, we show using lightcone bootstrap that there exist towers of "transverse derivative operators" (akin to the double twist operators) in the defect spectrum whose transverse twists[2] $\widehat{\Delta} - s$ approach some universal accumulation points in the spectrum for large values of s. Furthermore, the anomalous dimensions and OPE coefficients for these operators at finite s can be obtained as an expansion in $\frac{1}{s}$ (sec. 3.3). We also derive a Lorentzian inverse to the defect channel block expansion of the bulk two-point function of scalars (sec. 3.4). This inversion formula extracts the OPE data in the defect channel as analytic functions of s from a discontinuity in the causal correlator. When applied to the correlator with almost lightlike separated operators, it gives us analytic formulae in s that resum the large s expansions obtained from the lightcone bootstrap. Unlike in [74], we have not been able to establish any universal lower bound on s above which the inversion formula holds true. Apart from the review of lightcone bootstrap and OPE inversion in CFTs (without defects) in sec. 3.1, this chapter is based on original research presented in the author's publication [3] and contains some text from the same.

In chap. 4, we define Mellin amplitudes for the four-point function of spin-half fermions and the mixed (spin-half) fermion-scalar four-point function. These correlators are always expressed in a suitably chosen basis of tensor structures (sec. 4.2) and the Mellin amplitude is now a set of functions, with each component associated to one of the tensor structures in the basis (sec. 4.3). Restricting to the case of three spacetime dimensions, we carry out an analysis of the pole structure of these Mellin amplitudes which turns out to be significantly more involved than in the scalar case (sec. 4.4 and sec. 4.5). Furthermore, we compute the Mellin amplitudes associated with a few tree level Witten diagrams (sec. 4.6) and conformal Feynman integrals (sec. 4.7) and these demonstrate some generic properties of such Mellin amplitudes. The residues at the poles of the Mellin amplitude feature some kinematically fixed polynomials the knowledge of which is important to read off OPE coefficients from the residues. We leave to future work the derivation of the exact form of these polynomials and a subsequent application of this setup to bootstrapping fermionic CFTs. The research work discussed in this chapter (except the review of the Mellin formalism for scalar correlators in sec. 4.1) is based on and contains some excerpts from the author's publication [2].

In chap. 5, we generalize the work presented in [145] in the context of the AdS/CFT correspondence to reproduce the contribution of spinning operators to the OPE of scalars in the boundary CFT from the OPE of vertex operators in the worldsheet CFT of the dual string theory (sec. 5.2). As a natural consequence of this analysis, we obtain a set of relations obeyed by OPE coefficients in the boundary CFT and those in the worldsheet CFT (sec. 5.3). We generalize the analysis further to incorporate the contribution of a scalar to the OPE of conserved spin one currents in the boundary CFT thereby obtaining similar relations between OPE coefficients (sec. 5.4). Operator dimensions and OPE coefficients in certain sectors of the boundary CFT are sometimes subject to non-renormalization theorems owing to a high degree of symmetry (see for example [146–155]). In these cases, the relations between coupling constants in AdS supergravity and the OPE coefficients in the boundary CFT (that can be obtained from the evaluation of Witten diagrams) [144] can be supplemented with our relations between OPE coefficients to give a triangle connecting data that describe different regimes of the duality. This chapter is based on and has some overlap in text with the author's publication [1].

[2] $\widehat{\Delta}$ denotes the dimensions of local operators living on the conformal defect.

Chapter 2

CFT in $d > 2$ and defects

In this thesis, we shall discuss a few analytical methods to study CFTs in dimensions higher than two. The first of these approaches, see chap. 3, is about analytical approaches to conformal bootstrap in the context of a defect CFT where we shall make some non-trivial theory independent statements about the operator content of defect CFTs and the corresponding coefficients in the operator product expansion (OPE coefficients). We shall then shift our attention to the Mellin representation of conformal correlators and define and study Mellin amplitudes for correlators of fermionic operators in chap. 4. These Mellin amplitudes have analytic properties analogous to those of scattering amplitudes in QFT and these properties that we will look at again follow from the operator product expansion in CFT. Finally in chap. 5, we will study the operator product expansion itself, this time in the context of the AdS/CFT correspondence. We will review how the operator product expansion in a CFT can be understood from the physics on the dual worldsheet CFT and as a consequence, obtain some relations that corresponding OPE coefficients in these dual theories must satisfy.

It is evident that in all our explorations, a central role will be played by the operator product expansion in CFT and the data defining the theory, namely the conformal dimensions of operators and the corresponding OPE coefficients, and the general structure of correlation functions constrained by conformal symmetry. We would therefore like to set ourselves up for these discussions by reviewing the very basics of CFTs in dimensions higher than two.

Let us recall that CFTs are QFTs at the fixed points of the renormalization group flows. Interacting CFTs have a continuous mass spectrum and thus the excitations of these theories are not amenable to a particle interpretation. A famous example of such a CFT is the Wilson-Fisher fixed point in dimensions ranging from $d = 4 - \epsilon$ to $d = 2$. Although we start with this perturbative approach to defining CFTs, we shall be able to give a non-perturbative definition later on in the course of this chapter.

It is natural that we begin with a brief discussion on conformal transformations in sec. 2.1. From there we shall move on to the operator content of CFTs in sec. 2.2 and discuss the embedding formalism and conformal correlation functions in sec. 2.3. Thereafter, we shall review the operator product expansion in CFT in sec. 2.4 and discuss conformal blocks in sec. 2.5 thereby learning about the importance of OPE data in defining a CFT. Finally we shall review the condition of OPE associativity and how it can be exploited to obtain information on a CFT in sec. 2.6. We shall review each of these topics in the context of CFTs with and without defects.

This chapter is entirely a review and does not contain any original research work by the author. For further details on the topics touched upon here, one should refer to [156–159]. In the course of this chapter, we shall assume Euclidean signature for simplicity (except for sec. 2.3.4 where we shall temporarily shift to Minkowski signature in order to incorporate spinors) and hence the spacetime metric is $g^{\mu\nu} = \delta^{\mu\nu}$.

2.1 Conformal transformations

Conformal transformations are continuous coordinate transformations that leave the spacetime metric invariant upto a local scale factor. In other words, the Jacobian matrix of the coordinate transformation is proportional to a matrix of $SO(d)$.

$$\delta_{\mu\nu} \xrightarrow[\text{C.T.}]{} \delta_{\alpha\beta}\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\frac{\partial\tilde{x}^{\beta}}{\partial x^{\nu}} = \omega^2(x)\delta_{\mu\nu}\,. \tag{2.1}$$

Let us consider a generic infinitesimal local coordinate transformation,

$$\tilde{x}^{\mu} = x^{\mu} + \epsilon^{\mu}(x)\,. \tag{2.2}$$

Such an infinitesimal transformation would result in the following infinitesimal change in the metric $g^{\mu\nu}$,

$$\delta g^{\mu\nu} = \partial^{\mu}\epsilon^{\nu} + \partial^{\nu}\epsilon^{\mu}\,, \tag{2.3}$$

and this should be locally proportional to $\delta^{\mu\nu}$ for eq. (2.2) to correspond to an infinitesimal conformal transformation. It can be shown that this requirement fixes the infinitesimal variation ϵ^{μ} to be of the following generic form in dimensions $d > 2$,

$$\epsilon^{\mu}(x) = a_1^{\mu} + a_2 x^{\mu} + a_3^{\mu\nu}x_{\nu} + 2\,(a_4\cdot x)\,x^{\mu} - a_4^{\mu}x^2\,, \tag{2.4}$$

where a_1^{μ}, a_2, $a_3^{\mu\nu}$, a_4^{μ} are constant parameters and $a_3^{\mu\nu}$ is anti-symmetric in its indices. These parameters correspond to the following coordinate transformations:

- $\epsilon^{\mu}(x) = a_1$ is a translation which leaves the flat space metric invariant.
- $\epsilon^{\mu}(x) = a_2 x^{\mu}$ is a dilatation under which we have $\delta g^{\mu\nu} = 2a_2\delta^{\mu\nu}$. The corresponding finite transformation leaves the metric intact up to the scale factor of ω^2 as shown in eq. (2.1) with $\omega = e^{a_2}$.
- $\epsilon^{\mu}(x) = a_3^{\mu\nu}x_{\nu}$ with anti-symmetric $a_3^{\mu\nu}$ are orthogonal rotations (Lorentz transformations in Minkowski spacetime) that leave the flat metric invariant.
- $\epsilon^{\mu}(x) = 2\,(a_4\cdot x)\,x^{\mu} - a_4^{\mu}x^2$ is the infinitesimal special conformal transformation (SCT) producing a variation in the metric of the form $\delta g^{\mu\nu} = 4\,(a_4\cdot x)\,\delta^{\mu\nu}$. The change of scale in the metric resulting from the finite transformation with parameter a_4 is given by $\omega(x) = \left(1 - 2\,(a_4\cdot x) + a_4^2x^2\right)^{-1}$.

In two dimensions, the conformal group is larger as all holomorphic and anti-holomorphic transformations lead to a local rescaling of the metric. The transformations mentioned above form the subgroup of global conformal transformations in this case.

From the infinitesimal transformation in eq. (2.4), we can deduce the generators of conformal transformations and these can be represented with the following vector fields,

$$\begin{aligned}
\text{Translation:} \quad & P_\mu &=& \; i\partial_\mu \,, \\
\text{Dilatation:} \quad & D &=& \; i x^\mu \partial_\mu \,, \\
\text{Rotations:} \quad & M_{\mu\nu} &=& \; i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \,, \\
\text{SCT:} \quad & K_\mu &=& \; i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right) \,.
\end{aligned} \tag{2.5}$$

These generators define the conformal algebra in d dimensions with the following non-trivial commutation relations,

$$\begin{aligned}
[D, P_\mu] &= -iP_\mu \,, \\
[D, K_\mu] &= iK_\mu \,, \\
[P_\mu, K_\nu] &= 2i\left(\delta_{\mu\nu} D - M_{\mu\nu}\right) \,, \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\delta_{\mu\sigma} M_{\nu\rho} + \delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\nu} M_{\rho\sigma} - \delta_{\rho\sigma} M_{\mu\nu}\right) \,, \\
[M_{\mu\nu}, P_\rho] &= i\left(\delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu\right) \,, \\
[M_{\mu\nu}, K_\rho] &= i\left(\delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu\right) \,.
\end{aligned} \tag{2.6}$$

All other commutation relations vanish. It can be shown that the conformal algebra in d dimensional Euclidean spacetime is in fact isomorphic to the algebra of Lorentz transformations in $d+2$ dimensions $SO(d+1,1)$.

Let us now consider the presence of a p dimensional defect (co-dimension q) in the CFT in d dimensions. The presence of the defect breaks the d dimensional conformal symmetry as the symmetries of the theory are restricted to only those conformal transformations that leave the defect invariant. In the case of a flat defect, which is what we will consider in chap. 3, the residual symmetry is just $SO(p+1,1) \times SO(q)$ $(p+q=d)$.

With these preliminaries about conformal coordinate transformations, we are now ready to talk about operators in CFTs with and without defects.

2.2 Operators and states in a CFT

An important aspect to the characterization of a CFT is its spectrum of primary operators. A primary operator (acted on by a certain representation of the rotation group) is a local operator that transforms under conformal transformations in the following manner,

$$\phi(x) \underset{\text{CT}}{\longrightarrow} \tilde{\phi}(\tilde{x}) = \omega(x)^{-\Delta} R[\Lambda^{\mu\nu}]\phi(x) \,. \tag{2.7}$$

The CT in eq. (2.7) is assumed to be a composition of a rotation $\Lambda_{\mu\nu}$ and other conformal transformations (each of these maybe trivial). $R[\Lambda^{\mu\nu}]$ denotes the representation of the rotation group that acts on the operator $\phi(x)$ ($SO(d)$ indices of $\phi(x)$ have been suppressed).

$\omega(x)$ is the scale factor as shown in eq. (2.1). Δ is called the scaling dimension of $\phi(x)$ and this number along with the spin of $\phi(x)$ under the rotation group defines the operator. For example, a primary operator that is a vector has the following transformation,

$$\phi^{\mu}(x) \underset{\mathrm{CT}}{\longrightarrow} \tilde{\phi}^{\mu}(\tilde{x}) = \omega(x)^{-\Delta} \Lambda^{\mu\nu} \phi_{\nu}(x) \,. \tag{2.8}$$

If the CT in question is a dilatation $x \to ax$, $\omega(x) = a$. In fact, all operators in a CFT transform in the same manner under a dilatation. However the operator $\partial_{\mu}\phi(x)$ does not transform as per eq. (2.7) under special conformal transformations. Such operators which do not obey eq. (2.7), typically represented as derivatives of primaries, are called descendants. It can be proved that in unitary CFTs, all local operators are linear combinations of primaries and descendants.

Primary operators can also be defined with the property that at the origin, they commute with the generator of SCTs,

$$[K_{\mu}, \phi(0)] = 0 \,. \tag{2.9}$$

Other generators of the conformal algebra act on primary (and descendant) operators at the origin as follows,

$$\begin{aligned} [P_{\mu}, \phi(0)] &= -i\partial_{\mu}\phi(0) \,, \\ [D, \phi(0)] &= -i\Delta\phi(0) \,, \\ [M_{\mu\nu}, \phi(0)] &= -iS_{\mu\nu}\phi(0) \,. \end{aligned} \tag{2.10}$$

$S_{\mu\nu} = R\,[M^{\mu\nu}]$ is a finite dimensional matrix rotating the internal degrees of freedom of $\phi(0)$. Eq. (2.10) together with the commutation relations with the translation operator in eq. (2.6) give us the action of the conformal generators on a primary operator,

$$\begin{aligned} [K_{\mu}, \phi(x)] &= -i\left(2x_{\mu}\Delta + x^{\rho}S_{\rho\mu} + 2x_{\mu}\left(x^{\rho}\partial_{\rho} - x^{2}\partial_{\mu}\right)\right)\phi(x) \,, \\ [P_{\mu}, \phi(x)] &= -i\partial_{\mu}\phi(x) \,, \\ [D, \phi(x)] &= -i\left(\Delta + x^{\mu}\partial_{\mu}\right)\phi(x) \,, \\ [M_{\mu\nu}, \phi(x)] &= -i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} + S_{\mu\nu}\right)\phi(0) \,. \end{aligned} \tag{2.11}$$

Correlation functions in a CFT can be understood from a statistical viewpoint as weighted averages (path integrals) as well as from an algebraic viewpoint as inner products of states in a Hilbert space. The construction of Hilbert spaces in a QFT is associated with a choice of foliation of spacetime and a corresponding Hamiltonian. Each leaf of this foliation has its own Hilbert space and correlation functions are inner product of states on the same Hilbert space or on different Hilbert spaces connected by unitary evolution with an operator derived from the Hamiltonian. For example in the context of particle physics, QFTs are typically quantized with the leaves of the spacetime foliation being constant time surfaces. There is a Hilbert space of states defined at a given time on each leaf ie for each value of t. The P^0 operator is the Hamiltonian which generates the time evolution of the states.

CFTs in Euclidean spacetime can be quantized radially, which is to say that the Hilbert spaces of the theory live on concentric spheres centered at the origin (arbitrarily chosen).

The dilatation operator generates the evolution across these concentric spheres and thus it serves as the Hamiltonian. Alternatively, we can conformally map the spacetime to a cylinder $\mathbb{R} \times S^{d-1}$ and now the concentric spheres are mapped to cross-sections of the cylinder. The Hamiltonian generates translations along the length of the cylinder.

Operators and states in a CFT are characterised by their scaling dimension Δ and $SO(d)$ spin l.

$$\begin{aligned} D\,|\Delta, l\rangle &= i\Delta\,|\Delta, l\rangle\ , \\ M_{\mu\nu}\,|\Delta, l\rangle_i &\equiv [S_{\mu\nu}]_i^j\,|\Delta, l\rangle_j\ . \end{aligned} \tag{2.12}$$

i, j are $SO(d)$ indices of the state $|\Delta, l\rangle$ (which we usually suppress).

States on a given sphere, i.e. at a given radius, are created by inserting operators inside the sphere. Concretely, this amounts to evaluating a path integral over the interior of the sphere with these operator insertions. Insertion of no operators corresponds to the vacuum state $|0\rangle$ with vanishing dimension. An operator ϕ_Δ of dimension Δ inserted at the origin excites a state $|\Delta\rangle$. This can be seen from the commutation relations in eq. (2.10),

$$\begin{aligned} D\phi_\Delta(0)\,|0\rangle &= [D, \phi_\Delta(0)]\,|0\rangle + \phi_\Delta(0) D\,|0\rangle = i\Delta\phi_\Delta(0)\,|0\rangle\ , \\ \therefore \quad & \phi_\Delta(0)\,|0\rangle = |\Delta\rangle\ . \end{aligned} \tag{2.13}$$

Primary operators give rise to primary states and descendant operators give descendant states. It can be easily shown from eq. (2.9) that primary states are annihilated by K^μ. Inserting an operator at a point other than the origin can also excite a state living on the spheres surrounding it, however this state is no longer an eigenstate of the dilatation operator but is a superposition of these basis states.

$$\phi_\Delta(x)\,|0\rangle = e^{iPx}\phi_\Delta(0)e^{-iPx}\,|0\rangle = e^{iPx}\,|\Delta\rangle = \sum_{m=0}^{\infty} \frac{1}{m!}(Px)^m\,|\Delta\rangle\ , \tag{2.14}$$

where the states $P^m\,|\Delta\rangle$ are all descendants (ϕ_Δ could be a primary or a descendant).

Thus we have seen that corresponding to each local operator there exists a state in a CFT. This state could be a basis element (a primary or a descendant) of the Hilbert space, which is the case if a primary or descendant operator is inserted at the origin, or a superposition of basis states.

The converse also holds true in CFT. Given a state of dimension Δ, an operator of the same dimension can be constructed. Let us take a state $|\Delta, l\rangle$ and consider all correlation functions $\langle \phi_1(x_1)\cdots\phi_n(x_n)\,|\,\Delta, l\rangle$. A correlation function of this form can be evaluated as a path integral evaluated in the exterior of a ball centered around the origin (say) with the operator insertions $\phi_1(x_1)$ to $\phi_n(x_n)$ intact and a state with the dimension Δ and spin l excited on the boundary of this ball. Since an operator can be defined by the set of all its correlation functions with other operators, this effectively defines the operator insertion at the origin corresponding to the state $|\Delta, l\rangle$. In general, local operators can be defined corresponding to any state in a Hilbert space by using this construction and the scale symmetry.

The fact that there exists a state corresponding to each operator in a CFT and vice-versa is called the "operator-state correspondence" and is a key ingredient to proving the existence of a finite non-zero radius of convergence of the operator product expansion in CFT.

Let us conclude this discussion on operators and states by defining a conformal multiplet. Using the commutation relations in eq. (2.6), it is easy to show that the translation operator acts as a raising operator for scaling dimensions.

$$DP^{\mu}|\Delta\rangle = [D, P^{\mu}]|\Delta\rangle + P^{\mu}D|\Delta\rangle = iP^{\mu}|\Delta\rangle + i\Delta P^{\mu}|\Delta\rangle = i(\Delta+1)P^{\mu}|\Delta\rangle \ . \tag{2.15}$$

Similarly one can show that the generator of SCTs acts as a lowering operator for scaling dimensions as $[D, K^{\mu}] = -iK^{\mu}$.

$$DK^{\mu}|\Delta\rangle = i(\Delta-1)P^{\mu}|\Delta\rangle \ . \tag{2.16}$$

It can be shown that the requirement of unitarity in Lorentzian CFT (that translates to reflection positivity in Euclidean signature) puts a lower bound on the dimensions of operators popularly known as the unitarity bound. Therefore in a unitary CFT there must exist states that are annihilated by K_{μ} and thus by the operator-state correspondence there must exist primary operators in a unitary CFT.

Thus, starting from a primary state $|\Delta\rangle$, we can generate an infinite series of descendants by acting with the translation operator and the special conformal operator takes us down this ladder.

$$\begin{aligned} &|\Delta\rangle \underset{P^{\mu}}{\longrightarrow} |\Delta+1\rangle \underset{P^{\mu}}{\longrightarrow} |\Delta+2\rangle \underset{P^{\mu}}{\longrightarrow} \cdots \ , \\ |0\rangle \underset{K^{\mu}}{\longleftarrow} &|\Delta\rangle \underset{K^{\mu}}{\longleftarrow} |\Delta+1\rangle \underset{K^{\mu}}{\longleftarrow} |\Delta+2\rangle \underset{K^{\mu}}{\longleftarrow} \cdots \ . \end{aligned} \tag{2.17}$$

This construction is analogous to the construction of irreducible representations of $SU(2)$ starting with highest weight states. The vector space spanned by the primary state $|\Delta\rangle$ and its descendants $|\Delta+i\rangle$, $i > 0$ furnishes a representation of the conformal algebra and these basis states $|\Delta+i\rangle$, $i \geq 0$ are referred to as a conformal multiplet.

The discussion on local operators is not altered in the context of a CFT with a defect. We should remember that there are two CFTs in question here, one in the bulk and one on the defect and the discussion above holds individually to either of the two.

2.3 Conformal correlation functions

In this section, we shall see how symmetry constraints restricts the form of correlation functions in a CFT with or without defects. In the case of scalar operators, it is simple to implement the symmetry constraints on correlation functions in physical position space itself. However in the case of operators with spin, this becomes much more tedious. Therefore we use the embedding space formalism [160–165] that provides a neat and transparent way to constrain the form of correlation functions with the conformal symmetry and allows us to express and work with spinning conformal correlators in a more compact form. We shall describe the embedding formalism for symmetric traceless operators in sec. 2.3.1 and use it to deduce some conformal correlation functions in sec. 2.3.2. In sec. 2.3.3, we shall extend the discussion to correlators in defect CFTs. In sec. 2.3.4, we shall briefly review the embedding formalism for fermionic operators restricting to the case of three spacetime dimensions which we shall use in chap. 4.

The discussion on the embedding formalism for spinors in sec. 2.3.4 is excerpted from the author's publication [2].

2.3.1 Embedding space formalism

The embedding space formalism is based on the isomorphism of the conformal algebra in d dimensional Euclidean spacetime to the algebra of Lorentz transformations in $d+2$ dimensions $SO(d+1,1)$. Let us see how this isomorphism works explicitly. We consider the $d+2$ dimensional coordinates,

$$X^1,\, X^2,\, \cdots,\, X^d,\, X^{d+1},\, X^{d+2}\,, \tag{2.18}$$

with X^{d+2} being the timelike coordinate and define lightcone coordinates,

$$X^+ = X^{d+2} + X^{d+1}\,, \qquad X^- = X^{d+2} - X^{d+1}\,. \tag{2.19}$$

The line element is given in term of the lightcone coordinates as follows,

$$ds^2 = \sum_{\mu=1}^{d} \left(dX^i\right)^2 - dX^+ dX^-\,, \tag{2.20}$$

Let us now define the operators,

$$\begin{aligned} J^{\mu\nu} &= M^{\mu\nu}\,, \quad \mu,\nu \in \{1,2,\cdots,d\}\,, \\ J^{\mu+} &= P^{\mu}\,, \quad \mu \in \{1,2,\cdots,d\}\,, \\ J^{\mu-} &= K^{\mu}\,, \quad \mu \in \{1,2,\cdots,d\}\,, \\ J^{+-} &= D\,. \end{aligned} \tag{2.21}$$

It can be now shown that the operators J^{AB}, $A,B \in \{1,2,\cdots,d+2\}$ as defined in eq. (2.21) satisfy the commutation relations of $SO(d+1,1)$. We shall always use upper font English letters for indices in embedding space and lower font Greek or English letters for indices in physical space.

From eq. (2.5) and eq. (2.11), we see that a generic conformal transformation has a non-linear action in physical position space. However if we can embed physical space in $d+2$ dimensions, we can obtain a much desired linear action of the conformal transformations as these are just Lorentz transformations in this embedding space. This is the central idea behind the embedding space formalism.

Physical space can be embedded in $d+2$ dimensional Minkowski spacetime through the projective lightcone. Concretely, let us take the lightcone in $d+2$ dimensions,

$$X^2 = 0\,. \tag{2.22}$$

The lightcone is preserved by Lorentz transformations in $d+2$ dimensions (conformal transformations in d dimensions) and by restricting to this surface we already get rid of one of the two extra degrees of freedom.

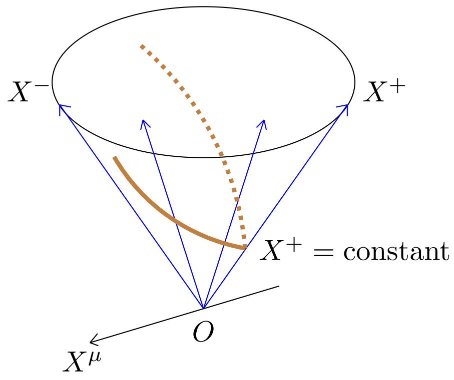


Figure 2.1: The projective lightcone: Physical d dimensional space is mapped to a section (in brown) of constant X^+ on the lightcone in $d+2$ dimensions. This is essentially the space of light rays (in blue) passing through the origin O.

We have an exact match in terms of number of degrees of freedom if we map physical space to the projective lightcone that is to the space of light rays on the lightcone as shown in fig. 2.1. Concretely, we can parametrize a section on the lightcone as,

$$X^A \equiv \left(X^\mu, X^+, X^-\right) = X^+\left(x^\mu, 1, x^2\right) , \quad \mu \in \{1, 2, \cdots, d\} \tag{2.23}$$

x^μ are coordinates in physical space. Choosing a constant value for X^+ would give us a section on the cone the induced metric on which is Euclidean. One can easily check that this parametrization satisfies the lightcone condition given by eq. (2.22). However it is not yet clear if the action of Lorentz transformations on X^A translates to an action of the corresponding conformal transformations on x^μ through the parametrization eq. (2.23). To verify this, let us consider a Lorentz transformation J^{AB} on the lightcone,

$$X^+\left(x^\mu, 1, x^2\right) \xrightarrow[J^{AB}]{} \tilde{X}^+\left(\tilde{x}^\mu, 1, \tilde{x}^2\right) . \tag{2.24}$$

The Lorentz transformation J^{AB} preserves the $d+2$ dimensional line element on the cone. Therefore for the d dimensional line elements, we must have,

$$\left(X^+\right)^2 \delta_{\mu\nu} dx^\mu dx^\nu = \left(\tilde{X}^+\right)^2 \delta_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu , \tag{2.25}$$

which implies that,

$$\tilde{g}_{\mu\nu} = \left(\frac{\tilde{X}^+}{X^+}\right)^2 \delta_{\mu\nu} . \tag{2.26}$$

Thus J^{AB} acts on x^μ in eq. (2.23) as a conformal transformation in d dimensions with $\frac{\tilde{X}^+}{X^+}$ being the factor $\omega(x)$ in eq. (2.1).

The next task is to embed primary operators $\phi(x)$ in physical space into operators $\Phi(X)$ defined on the lightcone. For simplicity, let us begin with scalars. $\Phi(X)$ transforms under Lorentz transformations in $d+2$ dimensions as follows,

$$\Phi(X) \xrightarrow[J^{AB}]{} \tilde{\Phi}(J\,X) = \Phi(X) . \tag{2.27}$$

We demand that $\Phi(X)$ be homogeneous in X as follows,

$$\Phi(a\,X) = a^{-\Delta}\Phi(X)\,, \tag{2.28}$$

Δ being the scaling dimension of $\phi(x)$, and that it equals the operator $\phi(x)$ on the projective section on the cone, that is,

$$\Phi(X)\Big|_{X=X^+(x^\mu,1,x^2)} = \phi(x)\,. \tag{2.29}$$

From eq. (2.26), eq. (2.28) and eq. (2.29), it is also evident that the action of a Lorentz transformation on $\Phi(X)$ defined on the cone translates to a conformal transformation acting on $\phi(x)$ as discussed in sec. 2.2,

$$\phi(x) \underset{\text{CT}}{\longrightarrow} \tilde{\phi}(\tilde{x}) = \omega(x)^{-\Delta}\phi(x)\,. \tag{2.30}$$

Now we move on to symmetric traceless representations of $SO(d)$ with non-zero integer spin. Let a spin l primary $\phi^{\mu_1\cdots\mu_l}(x)$ be embedded into a symmetric traceless operator on the lightcone $\Phi^{A_1\cdots A_l}(X)$. We impose the following transversality condition on $\Phi^{A_1\cdots A_l}(x)$.

$$X_{A_i}\Phi^{A_1\cdots A_l}(X) = 0\,, \tag{2.31}$$

and also the following homogeneity condition,

$$\Phi^{A_1\cdots A_l}(a\,X) = a^{-\Delta}\Phi^{A_1\cdots A_l}(X)\,, \tag{2.32}$$

Δ being the scaling dimension of $\phi^{\mu_1\cdots\mu_l}(x)$.

We prescribe the physical space operator $\phi^{\mu_1\cdots\mu_l}(x)$ to be obtained from the embedding space operator $\Phi^{A_1\cdots A_l}(X)$ as follows,

$$\phi^{\mu_1\cdots\mu_l}(x) = \Phi^{A_1\cdots A_l}(X)\prod_{i=1}^{l}\frac{\partial X_{A_i}}{\partial x_{\mu_i}}\Bigg|_{\{X_{A_i}\}=X^+\left(x_{\mu_i},1,x^2\right)}\,. \tag{2.33}$$

Note that since $X^2 = 0 \implies X^A\frac{\partial X_A}{\partial x_\mu} = 0$, we can always add terms proportional to X^{A_i} to $\Phi^{A_1\cdots A_l}(X)$ that are projected out by eq. (2.33). This redundancy together with the transversality condition in eq. (2.31) gives an agreement in the number of degrees of freedom when going from $\Phi^{A_1\cdots A_l}(X)$ to $\phi^{\mu_1\cdots\mu_l}(x)$. Furthermore, the projection in eq. (2.33) preserves the tracelessness when going of the embedding space operator in the physical space operator.

With an operator $\Phi^{A_1\cdots A_l}(X)$ thus defined, it can be shown that a Lorentz transformation (in $d+2$ dimensions) acting on $\Phi^{A_1\cdots A_l}(X)$ indeed translates to the corresponding conformal transformation (in d dimensions) acting on $\phi^{\mu_1\cdots\mu_l}(x)$,

$$\phi^{\mu_1\cdots\mu_l}(x) \underset{\text{CT}}{\longrightarrow} \tilde{\phi}^{\mu_1\cdots\mu_l}(\tilde{x}) = \omega(x)^{-\Delta}\Lambda^{\mu_1}_{\nu_1}\cdots\Lambda^{\mu_l}_{\nu_l}\phi^{\nu_1\cdots\nu_l}(x)\,. \tag{2.34}$$

Even in embedding space, spinning conformal correlators often come with a proliferation of indices which are tedious to handle. To tackle this issue an index free formalism was presented in [162]. The idea is to encode symmetric traceless tensors in polynomials using an auxiliary vector as follows,

$$\phi^{\mu_1\cdots\mu_l}(x) \to \phi^{(l)}(x,z) = z_{\mu_1\cdots\mu_l}\phi^{\mu_1\cdots\mu_l}(x)\,, \quad z^2 = 0\,. \tag{2.35}$$

The original tensor can be recovered from the polynomial applying the Todorov operator [166, 167] l times,

$$\begin{aligned} D^\mu &= \left(\frac{d-2}{2} + z\cdot\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z^\mu} + \frac{1}{2}z^\mu\frac{\partial^2}{\partial z\cdot\partial z}\,,\\ \phi^{\mu_1\cdots\mu_l}(x) &= \frac{1}{l!\left(\frac{d-2}{2}\right)_l}D^{\mu_1}\cdots D^{\mu_l}\phi^{(l)}(x,z)\,, \end{aligned} \tag{2.36}$$

$(a)_n$ is the Pochhammer symbol denoting $\frac{\Gamma(a+n)}{\Gamma(a)}$. The Todorov operator preserves the condition $z^2 = 0$.

The same construction can be repeated in embedding space.

$$\Phi^{A_1\cdots A_l}(X) \to \Phi^{(l)}(X,Z) = Z_{A_1}\cdots Z_{A_l}\Phi^{A_1\cdots A_l}(x)\,, \quad Z^2 = 0\,. \tag{2.37}$$

As in eq. (2.35), we restrict to $Z^2 = 0$ because of the tracelessness of $\Phi^{A_1\cdots A_l}(X)$. Transversality of $\Phi^{A_1\cdots A_l}(X)$ is encoded in the polynomial $\Phi^{(l)}(X,Z)$ by imposing $X\cdot Z = 0$. The tensor $\Phi^{A_1\cdots A_l}(X)$ can be obtained from $\Phi^{(l)}(X,Z)$ using the Todorov operator in $d+2$ dimensions as in eq. (2.36). When $\phi^{\mu_1\cdots\mu_l}(x)$ is embedded in $\Phi^{A_1\cdots A_l}(X)$, the equality of the corresponding polynomials $\phi^{(l)}(x,z)$ and $\Phi^{(l)}(X,Z)$ is ensured if $Z_{z,x} = \{z^\mu, 0, 2x\cdot z\}$, $\mu \in \{1,\cdots,d\}$. This is consistent with $Z^2 = 0$ and $X\cdot Z = 0$. Note that this index free formalism works for symmetric tensors that are not traceless as well and the generalization of $Z^2 = 0$ to this case is $Z^2 = z^2$.

Index free correlators can now be constructed in embedding space as functions of X_i, Z_i respecting the conditions of homogeneity, transversality, and $d+2$ dimensional Lorentz invariance. This automatically gives index free correlator in physical space as a polynomial from which one can obtain the correlator with indices using eq. (2.36). Alternatively, one can act on the index free correlator in embedding space with the Todorov operator in $d+2$ dimensions to get the embedding space correlator with indices. Using the projection in eq. (2.33), we can then obtain the correlator with indices in physical space.

2.3.2 Correlators of bosons

We shall now apply the formalism described above to deduce correlation functions of integer spin primaries. We shall limit the discussion to parity invariant correlators only. For convenience, we choose the lightcone section given by $X^+ = 1$.

Let us begin with the two-point function of scalar primaries $\langle\phi_1(x)\phi_2(y)\rangle$, the scaling dimension of the operator ϕ_i being Δ_i. $d+2$ dimensional Lorentz invariance and the homogeneity condition in eq. (2.28) fixes the two-point function in embedding space to be of the following form,

$$\langle\Phi_1(X)\Phi_2(Y)\rangle = K_{\Delta_1,0}\frac{\delta_{\Delta_1\Delta_2}}{(-2X\cdot Y)^{\Delta_1}}\,. \tag{2.38}$$

Δ_1 must equal Δ_2 for a non-zero two-point function as we cannot form scalars from X and Y individually since $X^2 = Y^2 = 0$. $K_{\Delta_1,0}$ is a numerical coefficient that can be set to 1. Using eq. (2.23) with $X^+ = 1$ gives us the correlator in physical space,

$$\langle \phi_1(x)\phi_2(y)\rangle = K_{\Delta_1,0}\frac{\delta_{\Delta_1\Delta_2}}{(x-y)^{2\Delta_1}}\,. \tag{2.39}$$

Next let's take up the three-point function of scalar primaries ϕ_i with dimensions Δ_i. Again, the conditions of $d+2$ dimensional Lorentz invariance and homogeneity fix the form of the three-point function to be as follows,

$$\langle \Phi_1(X_1)\Phi_2(X_2)\Phi_3(X_3)\rangle = \frac{\bar{\lambda}^{(0,0,0)}_{123}}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}X_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}}\,, \tag{2.40}$$

where we have used the notation $X_{ij} = -2X_i\cdot X_j$. $\bar{\lambda}^{(0,0,0)}_{123}$ is a numerical coefficient associated with the three-point function (the labels in subscript denote the scaling dimensions of the operators and the the ones in superscript tell the corresponding values of spin). Once we fix the numerical coefficient associated with the two-point function in eq. (2.38), there is no further freedom to choose $\bar{\lambda}^{(0,0,0)}_{123}$. Indeed this numerical factor is a piece of data that defines the CFT. In physical space, this three-point function takes the form,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\bar{\lambda}^{(0,0,0)}_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}}\,. \tag{2.41}$$

We have seen that the functional form of the two-point function and the three-point function of scalar primaries is completely fixed by conformal symmetry. However with four or higher number of operator insertions, we can construct conformal invariants out of the available coordinates. Therefore the functional form of the correlators of four or more operators is not fixed in a theory independent manner.

Let us consider a correlation function with n operator insertions. Using the available conformal symmetry we can fix some of the nd coordinates available. Subtracting the number of conformal generators from nd and adding the number of generators that stabilize the configuration achieved gives the number of degrees of freedom available and thus the number of independent conformal invariants that can be constructed. For $n = 2, 3$, conformal symmetries can fix all of the available coordinates. For $n = 4$, we have two degrees of freedom available for $d > 1$. The number of invariants that can be constructed from n points can be counted to be [168],

$$\frac{m(m-3)}{2} + d(n-m)\,, \qquad \text{where} \quad m = \mathrm{Min}\{n, d+2\}\,. \tag{2.42}$$

The most typical examples of conformal invariants are cross-ratios that have the form,

$$\frac{X_{ij}X_{kl}}{X_{ik}X_{jl}}\,, \tag{2.43}$$

and for the four-point function, the commonly used pair of cross-ratios is,

$$u = \frac{X_{12}X_{34}}{X_{13}X_{24}}\,, \qquad v = \frac{X_{14}X_{23}}{X_{13}X_{24}}\,. \tag{2.44}$$

There is no unique way to express the four-point function of scalar primaries. One of the commonly used forms consistent with $d+2$ dimensional Lorentz invariance and homogeneity is as follows,

$$\langle \Phi_1(X_1)\Phi_2(X_2)\Phi_3(X_3)\Phi_4(X_4)\rangle = \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}} \frac{\mathcal{A}(u,v)}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}} . \tag{2.45}$$

The form of $\mathcal{A}(u,v)$ cannot be fixed in a theory independent manner and as we shall see later, to study a particular CFT amounts to determining this function. The embedding space expression for the correlator can be easily translated to physical space using $X_{ij} \to (x_i - x_j)^2$.

Let us now apply the index free formalism in embedding space to a few spinning correlators. First we consider the two-point function of spin one operators. Demanding $d+2$ dimensional Lorentz invariance combined with homogeneity and transversality fixes this two-point function (upto a numerical coefficient) to be of the form,

$$\begin{aligned} \left\langle \Phi_1^{(1)}(X_1,Z_1)\Phi_2^{(1)}(X_2,Z_2)\right\rangle &= Z_1^A Z_2^B \left\langle \Phi_{1,A}(X_1)\Phi_{2,B}(X_2)\right\rangle , \\ &= K_{\Delta_1,1}\left(Z_1\cdot Z_2 - \frac{(Z_1\cdot X_2)\,(Z_2\cdot X_1)}{X_1\cdot X_2}\right)\frac{\delta_{\Delta_1\Delta_2}}{X_{12}^{\Delta_1}} . \end{aligned} \tag{2.46}$$

$K_{\Delta_1,1}$ is a numerical coefficient. Translating this expression to physical space as described in sec. 2.46 gives us the correlator (with indices) in physical position space,

$$\langle \phi_1^{\mu_1}(x_1)\phi_2^{\mu_2}(x_2)\rangle = K_{\Delta_1,1}\left(\delta^{\mu_1\mu_2} - \frac{2\,(x_1-x_2)^{\mu_1}\,(x_1-x_2)^{\mu_2}}{(x_1-x_2)^2}\right)\frac{\delta_{\Delta_1\Delta_2}}{(x_1-x_2)^{2\Delta_1}} . \tag{2.47}$$

The two-point function of (integer) spin l primaries can be similarly deduced to have the following form in embedding space [162],

$$\begin{aligned} \left\langle \Phi_1^{(l)}(X_1,Z_1)\Phi_2^{(l)}(X_2,Z_2)\right\rangle &= Z_1^{A_1}\cdots Z_1^{A_l} Z_2^{B_1}\cdots Z_2^{B_l}\left\langle \Phi_{1,A_1\cdots A_l}(X_1)\Phi_{2,B_1\cdots B_l}(X_2)\right\rangle , \\ &= K_{\Delta_1,l}\left(Z_1\cdot Z_2 - \frac{(Z_1\cdot X_2)\,(Z_2\cdot X_1)}{X_1\cdot X_2}\right)^l \frac{\delta_{\Delta_1\Delta_2}}{X_{12}^{\Delta_1}} . \end{aligned} \tag{2.48}$$

The corresponding position space expression [169] with indices can be obtained using operators Todorov as described in sec. 2.3.1.

$$\left\langle \phi_1^{\mu_1\cdots\mu_l}(x_1)\phi_2^{\nu_1\cdots\nu_l}(x_2)\right\rangle = K_{\Delta_1,l}\,\delta_{\Delta_1\Delta_2}\frac{J^{[\mu_1\,(\nu_1}\,(x_1-x_2)\cdots J^{\mu_l]\,\nu_l)}\,(x_1-x_2) - \text{traces}}{(x_1-x_2)^{2\Delta_1}} , \tag{2.49}$$

where we have used,

$$J^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2} . \tag{2.50}$$

The [] and () brackets in the indices of $J^{\mu\nu}$ in eq. (2.49) indicate symmetrization over the μ_i and the ν_i indices separately. The subtraction of traces is also to be carried out separately over the μ_i and ν_i indices.

The numerical coefficient $K_{\Delta,l}$ of the two-point functions can be set to one unless the operators involved have some other natural normalization. For example, the normalization of the stress tensor is fixed by the relevant Ward identity and hence the coefficient of the two-point function of the stress tensor is fixed and is related to the central charge which is a physically meaningful quantity.

Let us conclude our brief review of correlators of bosonic operators with a three-point function involving spinning operators, namely that of two scalars and one (integer) spin l primary. This correlator is fixed to be of the following form[162],

$$\left\langle \Phi_1 (X_1) \Phi_2 (X_2) \Phi_3^{(l)} (X_3, Z_3) \right\rangle = \bar{\lambda}_{123}^{(0,0,l)} \frac{((Z_3 \cdot X_1)(X_{23}) - (Z_3 \cdot X_2)(X_{13}))^l}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+l}{2}} X_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2+l}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+l}{2}}} . \tag{2.51}$$

In physical space, this correlator takes the following form (with Lorentz indices),

$$\left\langle \phi_1 (x_1) \phi_2 (x_2) \phi_3^{\mu_1 \cdots \mu_l} (x_3) \right\rangle = \bar{\lambda}_{123}^{(0,0,l)} \frac{(V^{\mu_1} (x_1, x_2, x_3) \cdots V^{\mu_l} (x_1, x_2, x_3) - \text{traces})}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+l} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2-l} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1-l}} , \tag{2.52}$$

where we have used the abbreviation $x_{ij} = x_i - x_j$, and the tensor structure is as follows,

$$V^{\mu} (x_1, x_2, x_3) = \frac{(x_1 - x_3)^{\mu}}{(x_1 - x_3)^2} - \frac{(x_2 - x_3)^{\mu}}{(x_2 - x_3)^2} . \tag{2.53}$$

2.3.3 Correlators in defect CFT

Let us now discuss the modifications to the embedding formalism necessary for a d dimensional CFT with a p dimensional defect with the codimension being q - for further details on the topic see [64].

The conformal symmetry of the ambient theory is reduced to $SO(p+1,1) \times SO(q)$ and the theory living on the Euclidean defect has a $SO(p+1,1)$ conformal symmetry and a $SO(q)$ global symmetry. In the ambient CFT, the discussion on embedding operators into operators in the $d+2$ dimensional embedding space and the index free formalism for tensor operators goes through without modifications. We should however note that we are now interested in invariants (or covariants) of $SO(p+1,1) \times SO(d)$. Therefore scalar quantities can now be built using two different inner products corresponding to $SO(p+1,1)$ and $SO(d)$ respectively.

$$X \bullet Y = X^A Y^B \eta_{AB} , \qquad X \circ Y = X^I Y^J \delta_{IJ} . \tag{2.54}$$

In eq. (2.54), A, B indices denote the directions parallel to the defect and I, J indices denote those orthogonal to the defect. The Levi-Civita tensor density can also be used to construct scalars when parity odd primaries are involved. The lightcone condition $X^2 = 0$ and the transversality condition $Z \cdot X = 0$ are still true for bulk insertions and therefore we have,

$$X \bullet X = -X \circ X , \qquad Z \bullet Z = -Z \circ Z , \qquad X \bullet Z = -X \circ Z . \tag{2.55}$$

For correlation functions of operators on the defect, we can just use the embedding formalism described above for the p dimensional CFT with some modifications to the index free formalism that we shall describe below. Defect operators transform under both the $SO(p)$ and $SO(q)$ rotations (the latter being a global transform for the defect operators). Correspondingly they are labeled by the scaling dimension $\widehat{\Delta}$ associated with the p dimensional conformal symmetry, and two spin quantum numbers: j (defining the representation of $SO(p)$ that acts on the defect operator) and s (defining the representation of $SO(q)$ that acts on the defect operator), with s being a global symmetry charge for the defect operators. We shall denote defect operators with a hat and the three quantum numbers as in $\widehat{\phi}_{\widehat{\Delta},j,s}$ (with the labels in subscript being suppressed at times). For index free expressions, we have to introduce two sets of auxilliary vectors for each point, z^a, Z^A with $a \in \{1, \cdots, p\}$, $A \in \{1, \cdots, p+2\}$, $\{Z^A\} = \{z^a, 0, 2x^a z_a\}$, and w^i, W^I with $i, I \in \{1, \cdots, q\}$, $\{W^I\} = \{w^i\}$. To recover the physical space tensors in the defect theory from the associated polynomials we have to use two different Todorov operators in each case, and similarly for the embedding space tensors.

$$
\begin{aligned}
D^a &= \left(\frac{p-2}{2} + z^b \frac{\partial}{\partial z^b}\right) \frac{\partial}{\partial z^a} + \frac{1}{2} z^a \frac{\partial^2}{\partial z^b \cdot \partial z_b}\,, \\
D^i &= \left(\frac{q-2}{2} + w^j \frac{\partial}{\partial w_j}\right) \frac{\partial}{\partial w^i} + \frac{1}{2} w^i \frac{\partial^2}{\partial w^j \cdot \partial w_j}\,.
\end{aligned} \tag{2.56}
$$

The reduced symmetry in the bulk, in particular the absence of translation symmetry in directions perpendicular to the defect, results in bulk operators having a non-zero one-point function. In the absence of a defect, the identity is the only operator to have a non-zero one-point function. It can be shown that the one-point function of a symmetric traceless spin l bulk operator is non-zero only for even l and is of the following form [64],

$$
\left\langle \Phi^{(l)}(X,Z) \right\rangle = \frac{a_\phi \left(Z \circ Z - \frac{(Z\circ X)^2}{X \circ X}\right)^{\frac{l}{2}}}{(X \circ X)^{\frac{\Delta_\phi}{2}}}\,, \tag{2.57}
$$

where a_ϕ is a numerical coefficient that is not fixed by conformal symmetry. Note that bulk operators are normalized to fix the coefficient of the two-point function far away from (or in the absence of) the defect and thus there is no further freedom to choose the one-point function coefficient a_ϕ. One-point functions of pseudotensors can be non-zero even for odd spin [64] as the one-point function can now be composed of the epsilon tensor. When the defect is of codimension one, the polynomial in the numerator is zero and thus in such cases only scalar operators have a non-zero one-point function. The physical space expression corresponding to eq. (2.57) can be obtained using the procedure involving the Todorov operators described above. The expression for scalar operators is simple and is given as follows,

$$
\langle \phi(x) \rangle = \frac{a_\phi}{|x_\perp|^{\Delta_\phi}}\,, \tag{2.58}
$$

where $|x_\perp|$ is the perpendicular distance of x from the defect.

The two-point function of bulk operators is no longer fixed by symmetry which is the case in the absence of any defects. We can indeed form two independent $SO(p+1,1) \times SO(q)$

invariants from two-points X_1 and X_2 which we can choose to be as follows [64],

$$\begin{aligned}\xi &= \frac{-2X_1 \cdot X_2}{\sqrt{(X_1 \circ X_1)\,(X_2 \circ X_2)}} = \frac{x_{12}^2}{|(x_1)_\perp||(x_2)_\perp|}\,,\\ \eta = \cos\theta &= \frac{X_1 \circ X_2}{\sqrt{(X_1 \circ X_1)\,(X_2 \circ X_2)}} = \frac{(x_{12})_i\,(x_{12})^i}{|(x_1)_\perp||(x_2)_\perp|}\,,\end{aligned} \tag{2.59}$$

where i runs over the directions perpendicular to the defect. ξ is the distance squared between the two bulk points normalized by their distances from the defect, and θ is the angle around the defect between the two bulk insertions. When the codimension is one, we only have one invariant which can be taken to be the first one in eq. (2.59). Note that the second invariant in eq. (2.59) can be interpreted as the cosine of an angle only in a Euclidean configuration. When the bulk theory is Lorentzian and the defect is spacelike, we shall rather use η as the invariant which can take arbitrary real values now.

Using the invariants in eq. (2.59), we can express the two-point function of scalar bulk primaries as follows,

$$\langle \Phi_1\,(X_1)\,\Phi_2\,(X_2)\rangle = \frac{\mathcal{A}\,(\xi,\theta)}{(X_1 \circ X_1)^{\frac{\Delta_1}{2}}\,(X_2 \circ X_2)^{\frac{\Delta_2}{2}}}\,, \tag{2.60}$$

where $\mathcal{A}\,(\xi,\theta)$ is a function that encodes dynamical information on the theory. The physical space expression is easily obtained by using $X_i \circ X_i = (x_i)_\perp^2$.

The correlators of defect operators $\widehat{\phi}$ are fixed by the p dimensional conformal symmetry as discussed in sec. 2.3.2 and the $SO(q)$ global symmetry. Let us for example consider the two-point function of two defect scalars of dimension $\widehat{\Delta}$ with $j = 0$ and any s [64]. This two-point function is diagonal in all the three quantum numbers $\widehat{\Delta}$, j, s and is given as follows in embedding space notation,

$$\left\langle \widehat{\Phi}_{\widehat{\Delta},0,s}\left(X_{1,\|},W_1\right)\widehat{\Phi}_{\widehat{\Delta},0,s}\left(X_{2,\|},W_2\right)\right\rangle = 2^s\,\frac{(W_1 \circ W_2)^s}{\left(-2X_{1,\|}\bullet X_{2,\|}\right)^{\widehat{\Delta}}}\,, \tag{2.61}$$

where $X_{i,\|}$ are embedding space points corresponding to points $x_{i,\|}$ on the defect. The two-point function above is fixed only up to an overall numerical coefficient but we have chosen this to be 2^s in eq. (2.61). This two-point function of defect scalars can be translated to the index-ful expression in physical space using Todorov operators as described previously. This gives us,

$$\left\langle \widehat{\phi}^{i_1\cdots i_s}_{\widehat{\Delta},0,s}\left(x_{1,\|}\right)\widehat{\phi}^{j_1\cdots j_s}_{\widehat{\Delta},0,s}\left(x_{2,\|}\right)\right\rangle = 2^s\,\frac{\mathcal{P}^{i_1\cdots i_s,j_1\cdots j_s}}{\left(x_{1,\|}-x_{2,\|}\right)^{2\widehat{\Delta}}}\,, \tag{2.62}$$

where i_k, j_l are $SO(q)$ indices carried by $\widehat{\phi}$ and $\mathcal{P}^{i_1\cdots i_s,j_1\cdots j_s}$ is an $SO(q)$ tensor symmetric and traceless in the two sets of $SO(q)$ indices and is made of the corresponding metric tensor δ^{ij}. It acts as a projector onto symmetric traceless tensors and can be obtained from the following expression [64, 162],

$$\mathcal{P}^{i_1\cdots i_s,j_1\cdots j_s} = \frac{1}{s!\left(\frac{q}{2}-s\right)_s}D^{i_1}\cdots D^{i_s}w^{j_1}\cdots w^{j_s}\,. \tag{2.63}$$

Furthermore we can have correlators of bulk operators with defect operators. The two-point function of a bulk operator and a defect operator is completely fixed by symmetries upto numerical coefficients. For example, the two-point function of a bulk scalar and a defect scalar with $s = 0$ with scaling dimensions Δ and $\widehat{\Delta}$ respectively is given by,

$$\left\langle \Phi\left(X\right) \widehat{\Phi}\left(Y_{\|}\right) \right\rangle = \frac{b_{\phi\widehat{\phi}}}{\left(-2X\bullet Y_{\|}\right)^{\widehat{\Delta}} \left(X \circ X\right)^{\frac{\Delta - \widehat{\Delta}}{2}}}\,, \tag{2.64}$$

which in physical space takes the following form,

$$\left\langle \phi\left(x\right) \widehat{\phi}\left(y_{\|}\right) \right\rangle = \frac{b_{\phi\widehat{\phi}}}{|x_{\|} - y_{\|}|^{2\widehat{\Delta}} |x_{\perp}|^{\Delta - \widehat{\Delta}}}\,. \tag{2.65}$$

The coefficient $b_{\phi\widehat{\phi}}$ carry physical information (as we shall discuss in sec. 2.4.1) and is not upto our choice as the bulk and defect operators are normalized using the respective two-point functions. The two-point function of a bulk scalar with a defect primary is non-zero only if $j = 0$ for the defect primary.

2.3.4 Embedding space formalism for spinors

So far, we have restricted our discussion of the embedding formalism to symmetric traceless representations of $SO(d)$. Let us now briefly review the embedding formalism for spinors in Minkowski spacetime [161, 163, 164] that we shall need in chap. 4. The nature of spinors is dimension dependent and we shall restrict our discussion to three dimensions with signature $-++$. The double cover of $SO(2,1)$ is isomorphic to $Sp(2,\mathbb{R})$ and the smallest fundamental representation is that of a real two dimensional vector space which describes Majorana fermions and the fundamental generators preserve a 2×2 symplectic tensor. We shall be following the conventions of [164]. In particular, gamma matrices (γ_μ in 3d and Γ_I in 5d) are chosen to be real.

For every spinor ψ^α (transforming in the fundamental representation), an auxiliary anti-fundamental spinor (primary of vanishing dimension) s_α is introduced, so that we can conveniently work with the scalar,

$$\psi(x, s) = s_\alpha \psi^\alpha(x). \tag{2.66}$$

The spinorial 5d conformal group is isomorphic to $Sp(4,\mathbb{R})$ (double cover of $SO(3,2)$) and the fundamental generators now preserve a 4×4 symplectic tensor. We embed $\psi^\alpha(x)$ into a 5d spinor on the lightcone $\Psi^I(X)$ (fundamental of $Sp(4,\mathbb{R})$), and again take an auxilliary anti-fundamental spinor S_I to define,

$$\Psi(X, S) = S_I \Psi^I\left(X\right), \qquad S_I = \sqrt{X^+} \begin{pmatrix} s_\alpha \\ -x^\alpha_{\ \beta} s^\beta \end{pmatrix}, \qquad x^\alpha_{\ \beta} = x^\mu \left(\gamma_\mu\right)^\alpha_{\ \beta}\,. \tag{2.67}$$

Transformation properties of $\Psi^I(X)$ under rotations and boosts dictate the precise manner in which 3d spinors are embedded into 5d spinors in general and then the transversality condition $S_I X^I_{\ J} = 0$ (where $X^I_{\ J} = X^A \left(\Gamma_A\right)^I_{\ J}$) fixes how S_I can be expressed in terms of s_α.

Further, the requirement that $\Psi(X,S)$ is a Lorentz scalar in 5d iff $\psi(x,s)$ is a scalar primary in 3d with dimension Δ fixes $\Psi(X,S)$ and $\psi(x,s)$ to be related in the following manner,

$$\Psi(X,S) = \frac{1}{(X^+)^\Delta}\psi(x,s). \tag{2.68}$$

$\Psi(X,S)$ has the homogeneity property,

$$\Psi(aX,bS) = a^{-\Delta-\frac{1}{2}}b\,\Psi(X,S). \tag{2.69}$$

The form of the correlators (along with the tensor structures) is then fixed by the requirements of 5d Lorentz invariance, homogeneity (2.69) and transversality.

In general, any real operator of spin l can be represented as $\phi^{\alpha_1\alpha_2\cdots\alpha_{2l}}$ where the α_i are fundamental indices of $Spin(2,1)$. Here $\phi^{\alpha_1\alpha_2\cdots\alpha_{2l}}$ is symmetric in all indices (this is possible only in three dimensions). As before, to work in an index free manner, we can introduce an auxiliary spinor s_α to form,

$$\phi(x,s) = s_{\alpha_1}\cdots s_{\alpha_{2l}}\phi^{\alpha_1\alpha_2\cdots\alpha_{2l}}(x). \tag{2.70}$$

An analogous construction gives the associated 5d operator $\Phi(X,S)$.

Application of this formalism to deduce the form of correlation functions and in particular the tensor structures has been discussed in sec. 4.2. For now, let us just look at the two-point function of spin half fermions. This is fixed to be of the following form,

$$\langle\Psi(X_1,S_1)\,\Psi(X_2,S_2)\rangle = i\frac{\langle S_1S_2\rangle}{X_{12}^{\Delta+\frac{1}{2}}}. \tag{2.71}$$

$X_{ij} = -2X_i\cdot X_j$ and $\langle S_1X_2X_3\cdots X_{k-1}S_k\rangle = (S_1)_I(X_2)^I_{\;J}\,(X_3)^J_{\;K}\cdots(X_{k-1})^K_{\;L}\,(S_k)^L$. In three dimensional physical space, the two-point function looks like,

$$\langle\psi^\alpha(x_1)\psi_\beta(x_2)\rangle = i\frac{(x_{12})^\alpha_{\;\beta}}{\left(x_{12}^2\right)^{\Delta+\frac{1}{2}}}. \tag{2.72}$$

2.4 Operator product expansion

Operator product expansion (OPE) in a QFT is an operator equation that allows us to replace the product of two local operators that are asymptotically close to each other with an infinite series of operators inserted in between them. Although this expansion in QFT is only asymptotic, the OPE in CFT [11–14] has a finite radius of convergence [15, 16] (see also [170, 171]) which makes it a very powerful tool to study CFT. The convergence of the OPE in CFT can be understood using the operator-state correspondence discussed in sec. 2.2.

Let us consider for example two scalar operators $\phi_1(0)$ and $\phi_2(x)$ and assume that there are no other operator insertions in a sphere centered at zero with radius $r > |x|$ - see fig. 2.2. Let $\phi_2(x)\phi_1(0)\,|0\rangle = |\Omega\rangle_r$ be the state excited by these operators insertions on the Hilbert space living at radius r. $|\Omega\rangle_r$ can be expanded in the basis of primary and descendant states

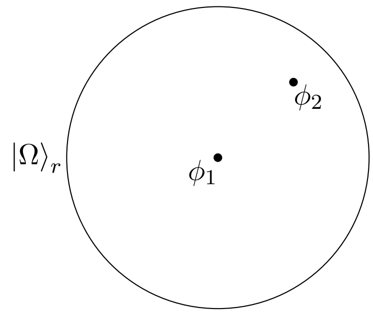


Figure 2.2: The operators $\phi_2\phi_1$ excite the state $|\Omega\rangle$ on the Hilbert space living on the surface of the sphere. $|\Omega\rangle$ can be expanded in a basis of primary and descendant states in this Hilbert space which in turn are excited by the corresponding primary and descendant operator insertions at the origin.

living at radius r. The operator-state correspondence then implies that we can write an operator equation of the following form,

$$\phi_2(x)\phi_1(0) = \sum_{\mathcal{O}} \frac{\lambda_{12\mathcal{O}}}{|x|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}} C^a_{\mathcal{O}}[x, \partial_y]\mathcal{O}_a(y)|_{y=0}\,. \tag{2.73}$$

The summation is over primary operators $\mathcal{O}$ including the identity operator. The contribution of descendants corresponding to each primary $\mathcal{O}$ is taken care of by $C_{\mathcal{O}}[x, \partial_y]$ which is an infinite sum of differential operators composed of ∂_y. The pre-factor $\frac{1}{|x|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}}$ and the exact form of the operator $C_{\mathcal{O}}[x, \partial_y]$ can be deduced by acting on the two sides of the expansion with the dilatation operator D and the momentum operator P^μ. Alternatively, we can expand $\phi_2(x)\phi_1(0)\mathcal{O}(z)$ using the OPE in eq. (2.73) which expresses it in terms of $C_{\mathcal{O}}[x, \partial_y]$ acting on the two-point function of $\mathcal{O}(z)$. We can do a term by term comparison of this expansion to the direct expansion of $\phi_2(x)\phi_1(0)\mathcal{O}(z)$ (whose form is fixed as discussed in sec. 2.3.2) around $x = 0$ to fix the form of the the OPE in eq. (2.73). $\lambda_{12\mathcal{O}}$ are called OPE coefficients and these are related to the numerical coefficients in the three-point and two-point functions. In particular, we can normalize the two-point function of $\mathcal{O}(z)$ to have unit coefficient and then the OPE coefficient $\lambda_{12\mathcal{O}}$ is simply equal to the three-point function coefficient $\bar{\lambda}_{12\mathcal{O}}$[1]. Note that in this case the OPE coefficient with the identity operator $\lambda_{\phi\phi I}$ is just 1. From now on, we shall assume that the two-point function of scalars are normalized to have a unit coefficient unless otherwise mentioned.

The index a in $C^a_{\mathcal{O}}[x, \partial_y]$ and $\mathcal{O}_a(y)$ stands for possible Lorentz indices that they maybe carrying. The operators contributing to the OPE in eq. (2.73) are integer spin operators as can be easily seen from the fact that the three-point function of two scalar operators with a fermionic operator is zero. The general features of the OPE discussed above generalize easily to the OPE of spinning primaries. The form of the OPE can be fixed entirely by symmetry considerations, although in practice it is easier to figure it out by expanding the three-point function using the OPE and comparing the result with a direct series expansion. The OPE in eq. (2.73) converges with the operators in the expansion inserted anywhere inside the sphere

[1]Note that we have suppressed the indices in the superscript of the three-point function coefficient $\bar{\lambda}_{12\mathcal{O}}$ (see eq. 2.40) and the OPE coefficient $\lambda_{12\mathcal{O}}$.

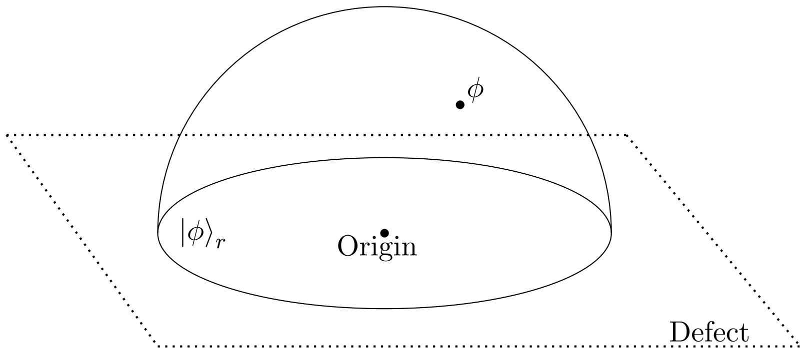


Figure 2.3: The bulk operator ϕ excite the state $|\phi\rangle_r$ on the Hilbert space living at radius r. This state can be expanded in terms of primary and descendant states which in turn can be excited by operator insertions inside the circle in the defect theory.

of radius r as long as the sphere separates ϕ_1 and ϕ_2 from other operator insertions.

The existence of a convergent OPE also implies that any conformal correlation function can be expressed in terms of differential operators acting on two-point functions with the form of the differential operators fixed entirely by the conformal symmetry. The only unknown pieces of information we need to write such an expression are the unknown ingredients in the OPE, namely the spectrum of primaries $\mathcal{O}$ contributing to the OPE of the operators involved and the corresponding OPE coefficients $\lambda_{ij\mathcal{O}}$. Since all correlation functions in a CFT can in principle be determined if we know the spectrum of the theory and OPE coefficients in a theory, this set of data (called CFT data) defines a CFT non-perturbatively without any reference to a Lagrangian description. Of course, CFT data are subject to consistency conditions (that we discuss in sec. 2.6) and any random set of numbers chosen as CFT data does not necessarily define a CFT.

2.4.1 Operator product expansion in the defect channel

In the presence of a p dimensional defect (with $p + q = d$), we can expand (as an operator equation) a bulk operator in terms of primaries and descendants of the defect theory. To see this let us consider a sphere of radius r centered on the defect that separates the bulk operator $\phi(x)$ from other operator insertions in the bulk. This operator excites a state $\phi(x)|0\rangle = |\phi\rangle_r$ on the Hilbert space living at radius r - see fig. 2.3. This state can now be expanded solely in the basis of primaries and descendants of the defect theory, thus giving us the defect channel OPE through the operator-state correspondence in the CFT living on the defect.

Analogous to the bulk OPE in eq. (2.73), the defect channel OPE is entirely fixed by the $SO(p+1,1) \times SO(q)$ symmetry. The defect channel expansion of a scalar bulk primary takes the following form [64],

$$\phi(x) = \sum_{\widehat{\mathcal{O}}} b_{\phi\widehat{\mathcal{O}}} \frac{x^{i_1} \cdots x^{i_s}}{|x_\perp|^{\Delta_\phi - \widehat{\Delta}_{\widehat{\mathcal{O}}} + s}} B_{\widehat{\mathcal{O}}}[\partial_{x_\parallel}] \widehat{\mathcal{O}}_{i_1 \cdots i_s}(x_\parallel) \,. \tag{2.74}$$

Recall that the hat in $\widehat{\mathcal{O}}$ indicates that it is an operator in the defect theory. $x_\perp$ denote the coordinates of x in directions perpendicular to the defect and $x_\parallel$ denotes coordinates of x in directions parallel to the defect. The sum over defect primaries $\widehat{\mathcal{O}}$ in eq. (2.74) is a double sum over the dimension $\widehat{\Delta}$ and transverse spin s of the defect primaries. Let us recall that defect primaries are labeled by three quantum numbers, the dimension $\widehat{\Delta}$, $SO(p)$ spin j and $SO(q)$ spin s. Only scalar ($SO(p)$ spin $j = 0$) defect primaries contribute to this expansion as the two-point function of a bulk scalar primary with defect primaries is zero for spinning defect primaries. $B_{\widehat{\mathcal{O}}}[\partial_{x_\parallel}]$ is a series of differential operators composed of derivatives in directions parallel to the defect that account for the contribution of descendants in the family of the defect primary $\widehat{\mathcal{O}}$.

The set of CFT data is now expanded in the presence of a defect. This now includes the spectrum and OPE coefficients (corresponding to the OPE of a pair of operators in the bulk or in the defect theories) of the bulk and the defect theories ($\{\Delta, \lambda_{ijk}\}$ and $\{\widehat{\Delta}, \lambda_{\widehat{ijk}}\}$ respectively) and also the one-point function coefficients a_i in the bulk theory, and the OPE coefficients appearing in the defect channel expansion of bulk primaries $b_{i\widehat{j}}$. Any arbitrarily chosen set of data does not constitute CFT data defining a consistent theory of a CFT with a defect and consistency conditions need to be imposed as we shall discuss in sec. 2.6.1.

2.5 Conformal blocks

As discussed in the previous section, conformal correlation functions can be calculated using the OPE provided we know the spectrum and the OPE coefficients. In fact, it is enough to consider four-point functions as all the dynamical data defining a theory is encoded in all the possible four-point functions in a theory. Computing a four-point function using the OPE involves the use of conformal blocks which are functions of conformal invariants that encode the contribution of a single conformal multiplet to the four-point function.

Let us consider the four-point function of identical scalar primaries $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$. We can expand the pairs $\phi(x_1)\phi(x_2)$ and $\phi(x_3)\phi(x_4)$ using the OPE in eq. (2.73). This gives us,

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \;=\; & \sum_{\mathcal{O}_1}\sum_{\mathcal{O}_2} \frac{\lambda_{\phi\phi\mathcal{O}_1}\lambda_{\phi\phi\mathcal{O}_2}}{|x_1-x_2|^{2\Delta_\phi-\Delta_{\mathcal{O}_1}}|x_3-x_4|^{2\Delta_\phi-\Delta_{\mathcal{O}_2}}} C^{a_1}_{\mathcal{O}_1}[x_1-x_2,\partial_{x_2}] \\ & C^{a_2}_{\mathcal{O}_2}[x_3-x_4,\partial_{x_4}]\left\langle(\mathcal{O}_1)_{a_1}(x_2)\,(\mathcal{O}_2)_{a_2}(x_4)\right\rangle . \qquad (2.75)\end{aligned}$$

a_1 and a_2 denote Lorentz indices that $\mathcal{O}_1$ and $\mathcal{O}_2$ may be carrying. Note that the identity operator contributes to the OPE of identical primaries and this contribution is included in the sum above.

Since the two-point function is diagonal, we must have $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ to have a non-zero contribution from the right hand side of eq. (2.75). This also gets rid of one of the summations as now we have a resultant sum over primaries $\mathcal{O}$. Note that the sum over primaries $\mathcal{O}$ is a double sum over scaling dimension and (integer) spin. Let us the denote the dimension and spin of intermediate primaries $\mathcal{O}$ with $\Delta_{\mathcal{O}}$ and l respectively. Using the two-point function of spinning bosonic operators in eq. (2.49), we get,

$$\begin{aligned}\langle\phi(x_1)\cdots\phi(x_4)\rangle &= \sum_{\mathcal{O}}\frac{K_{\Delta_{\mathcal{O}},l}\,\lambda^2_{\phi\phi\mathcal{O}}}{|x_1-x_2|^{2\Delta_\phi-\Delta_{\mathcal{O}}}|x_3-x_4|^{2\Delta_\phi-\Delta_{\mathcal{O}}}}C^{\mu_1\cdots\mu_l}_{\mathcal{O}}[x_1-x_2,\partial_{x_2}] \\ &\quad C^{\nu_1\cdots\nu_l}_{\mathcal{O}}[x_3-x_4,\partial_{x_4}]\,\frac{J_{[\mu_1\,(\nu_1}\,(x_2-x_4)\cdots J_{\mu_l]\,\nu_l)}\,(x_2-x_4)-\text{traces}}{(x_2-x_4)^{2\Delta_{\mathcal{O}}}}\,.\end{aligned} \tag{2.76}$$

Comparing eq. (2.76) with the representation of the four-point function in eq. (2.45), we see that the conformal amplitude $\mathcal{A}(u,v)$ corresponding to the four-point function is given by,

$$\begin{aligned}\mathcal{A}(u,v) = &\sum_{\mathcal{O}}K_{\Delta,l}\,\lambda^2_{\phi\phi\mathcal{O}}|x_1-x_2|^{\Delta_{\mathcal{O}}}|x_3-x_4|^{\Delta_{\mathcal{O}}}C^{\mu_1\cdots\mu_l}_{\mathcal{O}}[x_1-x_2,\partial_{x_2}]C^{\nu_1\cdots\nu_l}_{\mathcal{O}}[x_3-x_4,\partial_{x_4}] \\ &\frac{J_{[\mu_1\,(\nu_1}\,(x_2-x_4)\cdots J_{\mu_l]\,\nu_l)}\,(x_2-x_4)-\text{traces}}{(x_2-x_4)^{2\Delta_{\mathcal{O}}}}\,.\end{aligned} \tag{2.77}$$

The contribution of each conformal multiplet corresponding to a primary $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$ and spin l is encoded in the following function,

$$\begin{aligned}g_{\Delta_{\mathcal{O}},l}(u,v) &= K_{\Delta_{\mathcal{O}},l}|x_1-x_2|^{\Delta_{\mathcal{O}}}|x_3-x_4|^{\Delta_{\mathcal{O}}}C^{\mu_1\cdots\mu_l}_{\Delta_{\mathcal{O}}}[x_1-x_2,\partial_{x_2}]C^{\nu_1\cdots\nu_l}_{\Delta_{\mathcal{O}}}[x_3-x_4,\partial_{x_4}] \\ &\quad\frac{J_{[\mu_1\,(\nu_1}\,(x_2-x_4)\cdots J_{\mu_l]\,\nu_l)}\,(x_2-x_4)-\text{traces}}{(x_2-x_4)^{2\Delta_{\mathcal{O}}}}\,.\end{aligned} \tag{2.78}$$

$g_{\Delta,l}(u,v)$ is a conformal block for the four-point function of identical scalar primaries. Note that from the definition in eq. (2.78), it is clear that the conformal block carries no dynamical information in itself and is a function fixed entirely by the conformal symmetry. The four-point function of identical scalars can be thus be expanded in conformal blocks as follows,

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \frac{\mathcal{A}(u,v)}{|x_1-x_2|^{2\Delta_\phi}|x_3-x_4|^{2\Delta_\phi}}\,, \\ &= \frac{1}{|x_1-x_2|^{2\Delta_\phi}|x_3-x_4|^{2\Delta_\phi}}\left(1+\sum_{\mathcal{O}}\lambda^2_{\phi\phi\mathcal{O}}g_{\Delta_{\mathcal{O}},l}(u,v)\right)\end{aligned} \tag{2.79}$$

We have now extracted out the contribution of the identity operator and the coefficient to this term is 1 since we have chosen the two-point function of ϕ to have a unit coefficient. Eq. (2.79) makes it manifest once more that the four-point function of scalar primaries can be calculated just from the knowledge of the spectrum of integer spin primaries and the corresponding OPE coefficients.

The definition of the conformal block generalizes in an obvious manner to the case of non-identical scalar primaries. When we are considering a correlation function of spinning primaries, the correlation function has different components corresponding to different tensor structures and there are different conformal blocks corresponding to each of these different components. This is explained in further details in chap. 4 where we discuss fermionic correlation functions.

Conformal blocks can be calculated directly using the OPE as in eq. (2.78) although this is extremely tedious. Conformal blocks can also be derived by solving the eigenvalue equation of the quadratic Casimir of the conformal group [172]. When the scalars are identical as we have considered for simplicity (and are normalized such that the coefficient of the two-point function is 1), these blocks are given by [172, 173],

$$\begin{aligned} g^{(d=2)}_{\Delta,l}(z,\bar{z}) &= k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + k_{\Delta-l}(z)k_{\Delta+l}(\bar{z})\,, &(2.80)\\ g^{(d=4)}_{\Delta,l}(z,\bar{z}) &= \frac{z\bar{z}}{z-\bar{z}}\left(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z})\right)\,, &(2.81)\\ k_\beta(x) &= x^{\frac{\beta}{2}}\,{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta,x\right)\,. \end{aligned}$$

Here we have traded the standard cross-ratios u, v for the Dolan-Osborn invariants $z, \bar{z}$ that are related to u, v as follows,

$$u = z\bar{z}\,, \qquad v = (1-z)(1-\bar{z})\,. \tag{2.82}$$

Devising methods to calculate conformal blocks for generic conformal correlation functions is an important direction of research in CFT [68, 163, 164, 170, 172–188].

2.5.1 Conformal blocks in defect CFT

In the presence of a defect with codimension two or higher, the two-point function of bulk operators is an unknown function of two invariants as discussed in sec. 2.3.3 and can be expanded in conformal blocks. This can be done using the bulk channel OPE in eq. (2.73) or through the defect channel OPE of each bulk operator as given in eq. (2.74). Let us consider, for simplicity, the two-point function of identical scalar primaries in the bulk.

Bulk channel

The bulk channel block expansion follows directly from eq. (2.73) as shown below,

$$\langle\phi(x)\phi(y)\rangle = \sum_{\mathcal{O}} \frac{\lambda_{\phi\phi\mathcal{O}}}{|x-y|^{2\Delta_\phi-\Delta_\mathcal{O}}} C^{\mu_1\cdots\mu_l}_{\mathcal{O}}[x-y,\partial_y]\,\langle\mathcal{O}_{\mu_1\cdots\mu_l}(y)\rangle\,. \tag{2.83}$$

The contribution of the identity operator is incorporated in eq. (2.83). As discussed in sec. 2.3.3, only even spin primaries have a non-zero one-point function and thus contribute to the two-point function. Comparing eq. (2.83) to the form of the two-point function in eq. (2.60) gives us an expansion for the conformal amplitude $\mathcal{A}(\xi,\theta)$. Following the conventions of [64], we also extract a factor of $\xi^{-\Delta_\phi}$.

$$\mathcal{A}(\xi,\theta) = \xi^{-\Delta_\phi}\sum_{\Delta_\mathcal{O}}\sum_{l=0,2,\cdots} a_\mathcal{O}\lambda_{\phi\phi\mathcal{O}}\,\xi^{\Delta_\phi}\frac{(|x_\perp||y_\perp|)^{\Delta_\phi}}{|x-y|^{2\Delta_\phi-\Delta_\mathcal{O}}} C^{\mu_1\cdots\mu_l}_{\mathcal{O}}[x-y,\partial_y]\,\langle\mathcal{O}_{\mu_1\cdots\mu_l}(y)\rangle_k\,. \tag{2.84}$$

The subscript k in $\langle\mathcal{O}_{\mu_1\cdots\mu_l}(y)\rangle_k$ indicates that this is only the kinematical part of the one-point function as we have already extracted out the coefficient $a_\mathcal{O}$. In eq. (2.84), we have also replaced the sum over $\mathcal{O}$ with an explicit double sum over the scaling dimension and spin. We have not written the explicit expression for the one-point function $\langle \mathcal{O}_{\mu_1 \cdots \mu_l}(y)\rangle$ in physical space as this is tedious and involves writing the components with indices parallel to and perpendicular to the defect separately - see [64] for more details. ξ, θ are the invariants introduced in eq. (2.59) and r_i is the perpendicular distance of x_i from the defect.

The contribution of a single conformal multiplet in the bulk theory to the two-point conformal amplitude $\mathcal{A}(\xi, \theta)$ (often called the two-point function in a loose sense) constitutes a bulk channel conformal block for the two-point function.

$$g_{\Delta_{\mathcal{O}},l}(\xi,\theta) = \xi^{\Delta_\phi} \frac{(|x_\perp||y_\perp|)^{\Delta_\phi}}{|x-y|^{2\Delta_\phi - \Delta_{\mathcal{O}}}} C_{\mathcal{O}}^{\mu_1 \cdots \mu_l}[x-y, \partial_y] \langle \mathcal{O}_{\mu_1 \cdots \mu_l}(y) \rangle_k \, . \tag{2.85}$$

This function which is determined entirely by symmetries of the theory can be either computed directly using eq. (2.85) or as a eigenfunction to the quadratic Casimir of $SO(d+1,1)$. It is the entire conformal group that we have to consider and not the reduced symmetry group in the presence of a defect as the bulk channel OPE is a local property of the bulk operators transforming under $SO(d+1,1)$ and is not affected by the presence of a defect. This computation has been attempted in [64] where a recurrence relation has been provided for the lightcone expansion of the block. The lightcone limit of this block for the two-point function of identical bulk scalars is given as follows,

$$g_{\Delta,l}(\xi,\theta) \approx \xi^{\frac{\Delta - l}{2}} \sin^l \theta \, {}_2F_1\left(\frac{\Delta+l}{4}, \frac{\Delta+l}{4}, \frac{\Delta+l+1}{4}, \sin^2\theta\right) . \tag{2.86}$$

The expression for the block in eq. (2.86) assumes that the bulk scalars ϕ are normalized to have a unit two-point function coefficient and is consistent with the rest of our conventions for the correlators in the presence of a defect. A general closed form expression for the bulk channel blocks is not available. In the special case of codimension two defects, the Casimir equation can be mapped to the one for the four-point function blocks (in the absence of defects) and the bulk blocks for $d = 4, 6$ can be obtained from the expressions in eq. (2.80) and eq. (2.81) - see [64].

The two-point function of identical bulk scalars can thus be expanded in terms of the bulk channel blocks as follows,

$$\langle \phi(x)\phi(y)\rangle = \left(\frac{\xi^{-1}}{|x_\perp||y_\perp|}\right)^{\Delta_\phi} \left(1 + \sum_{\Delta_{\mathcal{O}}} \sum_{l=0,2,\cdots} a_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} \, g_{\Delta_{\mathcal{O}},l}(\xi,\theta)\right) . \tag{2.87}$$

We have now extracted the identity contribution to the two-point function in eq. (2.87) (which is the dominant contribution far away from the defect). Note that the bulk channel expansion in eq. (2.87) does not have positivity in the coefficients $a_{\mathcal{O}}\lambda_{\phi\phi\mathcal{O}}$. The bulk channel conformal blocks and the expansion of the two-point function in terms of the blocks generalizes in a simple manner when the external operators are not identical. We refer the reader to [64] for further details on the topic.

Defect channel

The two-point function of bulk scalars can also be expanded in blocks using the defect channel OPE in eq. (2.74). For the two-point function of identical scalars, we would have,

$$\langle\phi(x)\phi(y)\rangle = \sum_{\widehat{\mathcal{O}}} b^2_{\phi\widehat{\mathcal{O}}} \frac{x^{i_1}\cdots x^{i_s}}{|x_\perp|^{\Delta_\phi-\widehat{\Delta}_{\widehat{\mathcal{O}}}+s}} \frac{y^{j_1}\cdots y^{j_s}}{|y_\perp|^{\Delta_\phi-\widehat{\Delta}_{\widehat{\mathcal{O}}}+s}} B_{\widehat{\mathcal{O}}}[\partial_{x_\parallel}] B_{\widehat{\mathcal{O}}}[\partial_{y_\parallel}] \left\langle \widehat{\mathcal{O}}_{i_1\cdots i_s}(x_\parallel)\widehat{\mathcal{O}}_{j_1\cdots j_s}(y_\parallel)\right\rangle , \tag{2.88}$$

where we have already employed the fact that the two-point function of defect primaries is diagonal in its quantum numbers. The sum over defect primaries $\widehat{\mathcal{O}}$ in eq. (2.88) is a double sum over the scaling dimension $\widehat{\Delta}_{\widehat{\mathcal{O}}}$ and the transverse spin s as only defect primaries which are $SO(p)$ scalars contribute to the defect channel expansion of a bulk scalar.

Following eq. (2.88) and the two-point functions in eq. (2.60) and eq. (2.62), we define the defect channel blocks to be the following functions,

$$\widehat{g}_{\widehat{\Delta},s}(\xi,\theta) = 2^s\left(|x_\perp||y_\perp|\right)^{\Delta_\phi} \frac{x^{i_1}\cdots x^{i_s}}{|x_\perp|^{\Delta_\phi-\widehat{\Delta}_{\widehat{\mathcal{O}}}+s}} \frac{y^{j_1}\cdots y^{j_s}}{|y_\perp|^{\Delta_\phi-\widehat{\Delta}_{\widehat{\mathcal{O}}}+s}} B_{\widehat{\mathcal{O}}}[\partial_{x_\parallel}] B_{\widehat{\mathcal{O}}}[\partial_{y_\parallel}] \frac{\mathcal{P}_{i_1\cdots i_s,j_1\cdots j_s}}{\left(x_\parallel - y_\parallel\right)^{2\widehat{\Delta}_{\mathcal{O}}}} . \tag{2.89}$$

With these defect blocks as defined in eq. (2.89), the two-point function of identical bulk scalars can be expanded in the defect channel as follows,

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{\left(|x_\perp||y_\perp|\right)^{\Delta_\phi}} \sum_{\widehat{\mathcal{O}}} b^2_{\phi\widehat{\mathcal{O}}}\widehat{g}_{\widehat{\Delta},s}(\xi,\theta) . \tag{2.90}$$

These functions have been calculated in closed form [64] as eigenfunctions of the quadratic Casimir of the $SO(p+1,1)\times SO(q)$ symmetry group of the defect theory. Owing to the product form of the symmetry group, the Casimir eigenvalue equation can be separated in variables to two equations one corresponding to $SO(p+1,1)$ and the other corresponding to $SO(q)$. The defect block $\widehat{g}_{\widehat{\Delta},s}$ is given as a product of the angular part which solves the Casimir equation for $SO(q)$ and the radial part that solves the Casimir equation for $SO(p+1,1)$ as follows,

$$\widehat{g}_{\widehat{\Delta},s}(\chi,\theta) = \chi^{-\widehat{\Delta}}\, {}_2F_1\left(\frac{\widehat{\Delta}}{2}+\frac{1}{2},\frac{\widehat{\Delta}}{2},\widehat{\Delta}+1-\frac{p}{2},\frac{4}{\chi^2}\right)\binom{s+\frac{q}{2}-2}{\frac{q}{2}-2}^{-1} C_s^{\left(\frac{q}{2}-1\right)}(\cos\theta) , \tag{2.91}$$

where we have introduced the variable $\chi = \xi + 2\cos\theta$. The expression for the block presented above is consistent with the expansion of the bulk two-point function in eq. (2.90) given that away from the defect, we have assumed the bulk two-point function to have unit coefficient and that the coefficient of the defect two-point function is set to 2^s as shown in eq. (2.62). Note that the angular part of the block is given by a Gegenbauer polynomial only for integer s. In order to perform an analytic continuation in s as we wish to in chap. 3, we would have to use a different representation of the block.

Let us now present a form of the defect channel blocks in eq. (2.91) that we shall use in chap. 3. We change variables from $\chi, \eta = \cos\theta$ to the following,

$$\chi = r + \frac{1}{r} , \qquad \eta = \frac{1}{2}\left(w + \frac{1}{w}\right) . \tag{2.92}$$

Note that in Euclidean kinematics w is a phase while in the Lorentzian case it could be any real number. In these variables, the blocks in eq. (2.91) can be expressed in the following form using transformations of hypergeometric functions - see appendix. A for details.

$$\widehat{g}_{\widehat{\Delta},s}\left(r,w\right) = r^{\widehat{\Delta}} w^{-s}\, {}_2F_1\left(-s, \frac{q}{2}-1, 2-\frac{q}{2}-s, w^2\right)\, {}_2F_1\left(\widehat{\Delta}, \frac{p}{2}, \widehat{\Delta}+1-\frac{p}{2}, r^2\right). \tag{2.93}$$

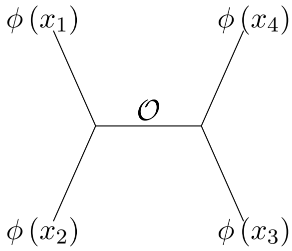
2.6 Crossing symmetry

In sec. 2.4, we discussed that all correlation functions in a CFT can in principle be calculated from the knowledge of the spectrum of the theory and it's OPE coefficients. Therefore this set of data called CFT data, subject to some consistency conditions, gives a non-perturbative definition of a CFT. In this section, we shall discuss such theory independent features of CFTs, and in particular the constraint of crossing symmetry given by the associativity of the OPE [14, 17, 189]. Let us consider a four-point function of identical scalar primaries $\langle\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\rangle$. Since the OPE is associative, we can take the OPE between different paris of operators as follows,

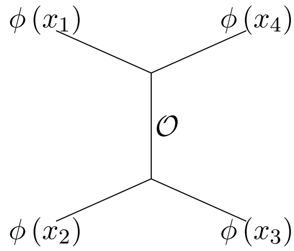
$$\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right) = \phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\,. \tag{2.94}$$

Taking the correlation function with $\phi\left(x_4\right)$ gives us block expansions for the four-point function $\langle\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\phi\left(x_4\right)\rangle$ in two different channels represented diagrammatically as follows,

$$\sum_{\mathcal{O}}$$



$$= \sum_{\mathcal{O}}$$



. (2.95)

Eq. (2.95) is the constraint of crossing symmetry on the four-point function of identical scalars. On the left hand side corresponds to a conformal block expansion in the $12-34$ channel - see eq. (2.79). On the right hand side we have the block expansion in the $14-23$ channel which is obtained from eq. (2.79) with the replacement $1 \leftrightarrow 3$ or $2 \leftrightarrow 4$. As crossing can be simply implemented by this replacement (which takes $u \leftrightarrow v$), the constraint of crossing symmetry on the four-point function of identical scalars of dimension Δ_ϕ is captured by the following exact equation,

$$\mathcal{A}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \mathcal{A}(v,u)\,. \tag{2.96}$$

Both sides of eq. (2.96) can be expanded in conformal blocks which gives us access to CFT data through this equation, as follows,

$$\left(v^{\Delta_\phi} - u^{\Delta_\phi}\right) + \sum_{\mathcal{O}} \lambda^2_{\phi\phi\mathcal{O}} \left(v^{\Delta_\phi} g_{\Delta_{\mathcal{O}},l}(u,v) - u^{\Delta_\phi} g_{\Delta_{\mathcal{O}},l}(v,u)\right) = 0\,. \tag{2.97}$$

The first term on the left hand side of eq. (2.97) is the contribution of the identity operator. The sum over $\mathcal{O}$ is a double sum over dimensions $\Delta_{\mathcal{O}}$ (non-zero) and spin l. Note that we do not have a term by term equality to zero in eq. (2.97). Using eq. (2.96) and generalizations thereof so as to obtain information on the spectrum and OPE coefficients in the theory is the central theme of the program of conformal bootstrap [17–33].

It can be shown that ensuring consistency with OPE associativity in all four-point functions in a theory ensures the same in all correlation functions in the theory. OPE associativity is not the only theory independent condition satisfied by CFTs. We are usually interested in studying unitary Lorentzian theories and unitarity (which translates to reflection positivity in Euclidean signature) puts lower bounds on operator dimensions as follows [66–70],

$$\begin{aligned} \Delta \geq&\quad \tfrac{d-2}{2}\,, & l &= 0\,, \\ \Delta \geq&\quad d+l-2\,, & l &> 0\,, \end{aligned} \tag{2.98}$$

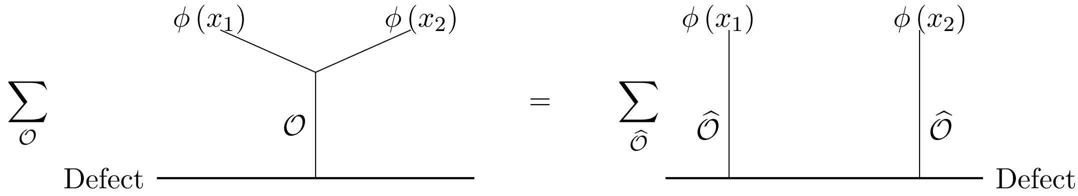
and we also have the identity operator with a vanishing dimension.

Primaries that saturate the unitarity bound have a null state in the corresponding conformal multiplet. Consequently, a scalar saturating the bound is a free scalar and an integer spin operator is a conserved current. Examples of these are the stress tensor with dimension d and spin two and conserved currents from global symmetries with dimension $d-1$ and spin one. Existence of the stress tensor and global symmetry currents is also a condition on CFT data following from the assumption of locality of the theory. Existence of currents with $l \geq 3$ in a theory in $d \geq 3$ also imply that the theory is free [190, 191].

Constraints on CFTs are also imposed by causality - see for example [192–195]. Furthermore one can consider CFTs on manifolds not conformally equivalent to flat space or theories that admit extended objects like boundaries, defects, etc. In this case, set of CFT data is expanded as we have discussed in sec. 2.4.1. These additional data come with some extra consistency conditions as we shall discuss now in the context of defects.

2.6.1 Crossing symmetry in defect CFT

Crossing symmetry of the four-point function, as discussed in the previous section, can be used to constrain CFT data for the theory living on the defect. Furthermore we also have to ensure OPE associativity in the two-point functions of bulk operators that give new constraints on data related to the two theories and their interaction. Let us consider the two-point function of identical bulk scalars $\langle \phi(x_1)\, \phi(x_2) \rangle$. We can expand this in conformal blocks in the bulk channel as shown in eq. (2.87) or in the defect channel as shown in eq. (2.90). Equality of these two expansions is the statement of crossing symmetry for the bulk two-point function.



(2.99)

On the left hand side of eq. (2.99) we have a summation over bulk primaries contributing to the OPE of $\phi(x_1)\,\phi(x_2)$ that have a non-zero one-point function in the presence of the defect, while on the right hand side we have a sum over defect primaries contributing to the defect channel OPE of ϕ. Concretely, we can write the following crossing equation from eq. (2.87) and eq. (2.90),

$$\xi^{-\Delta_\phi}\left(1+\sum_{\Delta_{\mathcal{O}}}\sum_{l=0,2,\cdots} a_{\mathcal{O}}\lambda_{\phi\phi\mathcal{O}}\; g_{\Delta_{\mathcal{O}},l}(\xi,\theta)\right)=\sum_{\widehat{\Delta}_{\widehat{\mathcal{O}}}}\sum_{s} b^2_{\phi\widehat{\mathcal{O}}}\widehat{g}_{\widehat{\Delta}_{\widehat{\mathcal{O}}},s}(\xi,\theta)\,. \tag{2.100}$$

As mentioned before, defect primaries with non-zero $SO(p)$ spin j do not contribute to the defect channel expansion on the right hand side of eq. (2.100). The crossing equation in eq. (2.100) is an important tool to study CFTs with defects ($q \geq 2$), boundaries and interfaces, [43, 44, 48, 53, 56, 63, 65, 71–73].

One important difference in the spectrum of the theory living on the defect from the bulk theory is that the defect theory does not have a conserved stress tensor of its own as this theory is interacting with the bulk theory. A quick way to verify this is as follows. If the defect theory did have its own conserved stress tensor it would be possible to translate the defect insertion of the bulk defect two-point function in eq. (2.65) without affecting the bulk insertion and still leave the correlation function invariant. This however is not possible as evident from eq. (2.64) or eq. (2.65).

The defect spectrum does however feature a new protected operator. This is the displacement operator $\widehat{D}^i$ which has a dimension $\widehat{\Delta} = p + 1$ and transforms as a vector under the $SO(q)$ rotations and hence has $s = 1$. The displacement operator appears in the stress tensor Ward identities and encodes the breaking of translation symmetry in directions orthogonal to the defect,

$$\partial_\mu T^{\mu i}\left(x^a_\parallel, x^j_\perp\right) = \widehat{D}^i\left(x^a_\parallel\right)\delta^{(q)}\left(x^j_\perp\right)\,. \tag{2.101}$$

Note that much like the stress tensor, the normalization of the displacement operator is fixed by the Ward identity and hence the coefficient of the two-point function of the stress tensor cannot be chosen and is a property of the defect theory. For the trivial defect, this coefficient is zero.

In this chapter, we reviewed some basics on CFTs in dimensions greater than two including conformal symmetries and how they constrain the form of conformal correlation functions, the convergent operator product expansion in CFT and how that enables us to reconstruct all correlation functions in the theory just from a consistent set of data on the spectrum and the OPE coefficients. Along the way, we also discussed how all of these aspects of CFT generalize to the case when there is a conformal defect in the theory that reduces the symmetry enjoyed by the ambient CFT.

Having familiarized ourselves with the basic concepts mentioned above and the importance of CFT data in defining a CFT non-perturbatively, let us now move on to some original research work pertaining to conformal correlators and in particular the CFT data that they carry.

Chapter 3

Universality in defect CFT at large transverse spin

In this chapter, we shall discuss the universality in the spectrum of a defect CFT at large transverse spin s. This line of inquiry derives inspiration from the large spin expansion in CFT [27, 28] and the Lorentzian inverse to the OPE derived by Caron-Huot [74]. We begin the chapter by acquainting ourselves with these ideas in section 3.1. Thereafter, we move on to present the case of defect CFT. By utilizing the crossing symmetry between the bulk and defect channel expansions of the two-point function of identical bulk scalars, we shall show that the spectrum of any defect CFT has universal accumulation points at large s. More precisely, we shall show that if there exists a scalar primary operator of dimension Δ_ϕ in the spectrum of the ambient CFT, the defect spectrum contains primaries with transverse twist $\widehat{\tau} = \widehat{\Delta} - s$ taking the following values,

$$\widehat{\tau} \simeq \Delta_\phi + 2m\,, \quad m \in \mathbb{N}\,, \quad s \to \infty\,. \tag{3.1}$$

We can also set up a perturbation theory around this universal limit and compute the defect CFT data in a $1/s$ expansion if the bulk CFT data is available.

The defect primaries in eq. (3.1) are obtained from the decomposition of the conformal family of a scalar bulk primary localized on the defect in representations of $SO(p+1,1) \times SO(q-1,1)$ symmetry group of the defect CFT. They can be schematically denoted as $\partial_i^s(\partial^j\partial_j)^m\phi$, where i, j denote directions orthogonal to the defect. It must be noted that it does not make sense to write such an operator in a generic strongly interacting theory away from any kinematic limits. The theory must admit a small parameter in order that such an operator exists. For example, operators of this kind are also present in the defect spectrum of large N theories. The relevant small parameter in this case is $\frac{1}{N}$. The non-trivial statement that we shall explain in this chapter is that any defect CFT admits a large s expansion and that an operator of the schematic form $\partial_i^s(\partial^j\partial_j)^m\phi$ is well-defined for asymptotically large values of s. We shall refer to such operators as *transverse derivative operators*. We shall show the existence of the transverse derivative operators and derive $\frac{1}{s}$ corrections using lightcone bootstrap techniques in sec. 3.3.

We shall then go further and derive a Lorentzian inversion formula for the bulk-defect

OPE (3.82) which extracts the scaling dimensions of defect operators and bulk-to-defect OPE coefficients as analytic functions of s from a discontinuity in the (Lorentzian) bulk two-point function. In the context of large s transverse derivative operators, it gives us analytic formulae in s that resum the $\frac{1}{s}$ expansions obtained from the lightcone bootstrap. This inversion formula and thus the analyticity in s of the defect CFT data is valid only for s larger than a certain minimum $s_\star$. A theory independent upper bound to $s_\star$ has not been obtained so far. We shall discuss this inversion formula in sec. 3.4.

In sec. 3.5 we understand the general results in the context of a free theory with a defect and in sec. 3.6 we apply the general discussion on lightcone bootstrap and the inversion formula to the twist defect in the 3D Ising CFT. If we map the theory on to a cylinder, operators of the form $\partial_i^s(\partial^j\partial_j)^m\phi$ correspond to states associated with particles rotating around the defect. The anomalous dimensions of the operators can then be calculated as the potential energy of these rotating particles due to interactions with the defect, exactly as done by Alday and Maldacena in [196]. Based on such qualitative arguments alone, we can obtain the general form of the large s corrections to the dimensions as predicted by lightcone bootstrap and OPE inversion [3]. We shall however not go into the details of these arguments in this chapter.

This chapter, except for section 3.1, is based on and contains excerpts from the author's publication [3].

3.1 Inspiration

Owing to the presence of the extra conformal symmetries in a CFT over a generic QFT, we can hope to study the space of all CFTs and classify them using symmetry and consistency conditions alone. This is the goal of the programme of conformal bootstrap, which is a non-perturbative approach to study CFT. The modern numerical bootstrap technique [197] has seen a significant amount of success to this end, for example in the context of the 3d Ising CFT [33]. In order to make such progress analytically for any CFT, it is important that the theory admits a small parameter in which we can set up a perturbation theory. It is not obvious that a generic strongly coupled CFT should admit such a small parameter.

Herein lies the great importance of the result [27, 28] that every CFT admits a large spin (l) expansion and that this regime is accessible by bringing operators to lightlike separation in Lorentzian correlators. Let us consider any CFT in $d > 2$ and let us assume that there exist primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with twists τ_1 and τ_2 respectively in the spectrum of this CFT. Then it can be shown that for each $m \in \mathbb{N}$, there exists an infinite tower of primaries whose twists approach the value $\tau_1 + \tau_2 + 2m$ as $l \to \infty$. The operators with this limiting value of twist can be schematically denoted as $\mathcal{O}_1\partial_{\mu_1}\cdots\partial_{\mu_l}\left(\partial^2\right)^m\mathcal{O}_2$ and are referred to as "double twist operators". Note that in a generic strongly interacting theory and away from the large spin limit, there may not be any operator with twist $\tau_1 + \tau_2 + 2m$ for any $m \in \mathbb{N}$ and as such $\mathcal{O}_1\partial_{\mu_1}\cdots\partial_{\mu_l}\left(\partial^2\right)^m\mathcal{O}_2$[1] is not a well-defined operator. This universality in the spectrum (and also OPE coefficients) of CFTs are consequences of crossing symmetry and

[1]Note that the operator $\mathcal{O}_1\partial_{\mu_1}\cdots\partial_{\mu_l}\left(\partial^2\right)^m\mathcal{O}_2$ is not generically a primary operator but it defines an equivalence class of operators corresponding to a primary.

unitarity.

As has been emphasized on previously, a CFT is completely defined by the CFT data and hence a very important aspect of research in CFT is to obtain anomalous dimensions and OPE coefficients upto a high degree of precision. To this end, one can now set up a perturbation theory around this universal $l \to \infty$ limit and calculate finite l corrections to the dimensions and OPE coefficients. These corrections are accessesed through a double lightcone limit of the Lorentzian four-point function. While one lightcone limit enables us to zoom onto the contrbution of the large spin double twist operators, the other one controls the double twist tower (specified by m) that we focus on. Solving the crossing symmetry equation in this configuration gives the desired expansion of anomalous dimensions and OPE coefficients in a series in $\frac{1}{l}$. The large spin expansion thus obtained through lightcone bootstrap is only asymptotic [30]. It must be noted that we do not gain any information on individual OPE coefficients in this manner and the relations obtained are true only in an averaged sense. There has been considerable progress in studying this expansion and applying the lightcone bootstrap techniques [27–31, 198–203].

The existence of the double twist operators in a CFT can be understood in an even more elegant fashion in the light of the Lorentzian inverse to the OPE derived by Caron-Huot [74] (also see [75]). Lorentzian correlators have branch cuts when operators are causally separated. Caron-Huot's formula extracts the OPE data from the corresponding (double) discontinuity in the correlator such that dimensions of operators and OPE coefficients are now expressed as analytic functions of spin. In Caron-Huot's own words [74], this establishes "the phenomenon of analyticity in spin in conformal field theories".

A crucial ingredient of Caron-Huot's formula is the behavior of correlation functions in the Regge limit. We shall now take a short detour and discuss a toy example presented by Caron-Huot [74] which neatly illustrates the fundamental ideas behind his inversion formula and the importance of the behavior in the Regge limit. Let us consider a Taylor series expansion,

$$f(E) = \sum_{J=0}^{\infty} f_J E^J \,, \tag{3.2}$$

and assume that $f(E)$ is analytic except for branch cuts at real $|E| \geq 1$ (depicted by the red lines in fig. 3.1), and that $\left|\frac{f(E)}{E}\right|$ is bounded at infinity. We can simply invert this relation using Cauchy's integral formula and obtain,

$$f_J = \frac{1}{2\pi i} \oint_{|E|<1} dE\, f(E) E^{-J-1}. \tag{3.3}$$

As shown in the fig. 3.1, we can now deform the contour of integration and drop off the arcs at infinity to obtain two analytic formulae in J for f_J,

$$\begin{aligned} f(J) &= \frac{1}{2\pi} \int_1^{\infty} \frac{dE}{E^{J+1}} \left[Disc\, f(E) + (-1)^J Disc\, f(-E) \right], & (3.4) \\ Disc\, f(E) &= \lim_{\epsilon \to 0} -i \left[f(E(1+i\epsilon)) - f(E(1-i\epsilon)) \right]. & (3.5) \end{aligned}$$

Although any single term in the Taylor expansion (3.2) is analytic in itself and thus does not contribute to the discontinuity, we can still extract the individual f_J from the discontinuity.

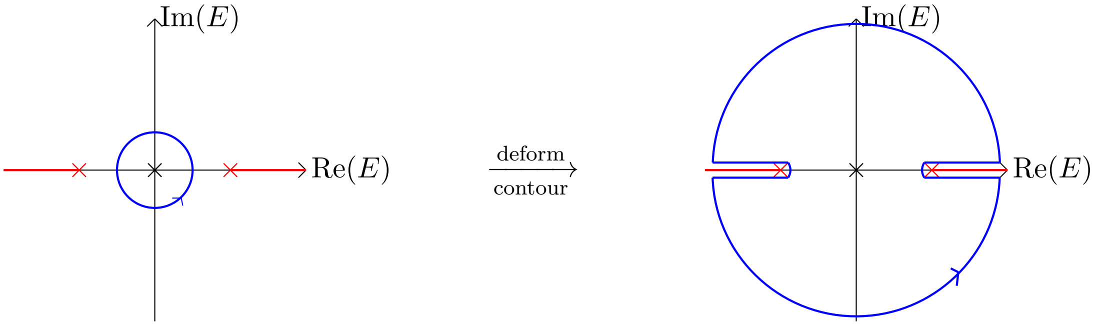


Figure 3.1: Obtaining f_J as analytic functions of J using boundedness of $f(E)$ at infinity.

This seeminly paradoxical phenomenon is made possible precisely by the behavior of the function at infinity. The assumption of boundedness at infinity prevents us from being able to change finitely many of the f_J without changing the others. In other words, the high energy Regge limit of $f(E)$ is linked with the fact that the f_J have an analytic structure to them as evident from (3.4), (3.5).

In CFT, we can think of the correlator as the analogue of $f(E)$, the conformal block expansion as the analogue of the Taylor expansion (3.2) and the OPE coefficients as the analogue of f_J. Caron-Huot's formula is analogous to (3.4) and extracts the OPE data as analytic functions of spin. Once again, the validity of the formula, which he establishes in general for all values of spin down to two, hinges on the appropriate behavior of the correlation functions in the Regge limit. We learn that although OPE coefficients in a CFT are independent of each other, they are still organized into an analytic structure.

When applied to the context of the double twist operators and the large spin expansion, this OPE inversion formula actually gives results that resum the expansions obtainable from lightcone bootstrap, thus giving us control over individual OPE coefficients as opposed to only weighted averages that was offered to us by the lightcone bootstrap. We realize that the spectrum of double twist operators in a CFT is organized into Regge trajectories that tend to the accumulation points $\tau = \tau_1 + \tau_2 + 2m$ for $m \in \mathbb{N}$ for any generic couple of primaries with twists τ_1 and τ_2 present in the spectrum. The contribution of individual Regge trajectories to the crossing equation is enhanced and isolated when pairs of operators become light-like separated which is why the double twist operators and the associated large spin expansion were revealed to us by the lightcone bootstrap.

Being inspired by these developments, we will make an attempt at lightcone bootstrap in the context of defect CFTs and discover the existence of the transverse derivative operators and the associated expansion in large s which is the transverse spin. Furthermore we will derive a Lorentzian inverse to the bulk-defect OPE which will establish the existence of $\widehat{\Delta}(s)$ trajectories. The inversion formula and thus the analyticity in s is valid only for $s > s_*$. So far we have not been able to obtain a theory independent upper bound for s_* unlike in the case of CFT.

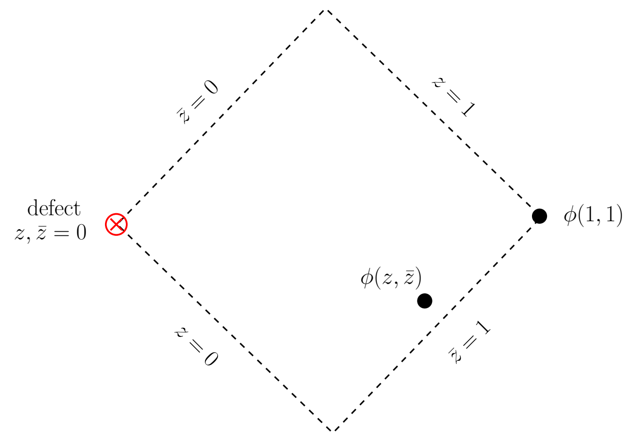


Figure 3.2: The configuration of the operators in $\langle\phi(x_1)\phi(x_2)\rangle$. We show a two dimensional plane, transverse to the defect, where the two operators lie. The defect is spacelike, it intersects the plane at the origin, and the two operators are placed at $(1,1)$ and $(z,\bar{z})$ respectively.

3.2 The Setup

Let us now establish the notation and discuss the setup for our analysis. We consider a flat defect of codimension q in d spacetime dimensions. We shall also denote the dimension of the defect by p, *i.e.*, $p+q=d$. We shall consider a Lorentzian CFT and assume the defect to be spacelike (unless mentioned otherwise) and therefore the symmetries of the theory are $SO(p+1,1)\times SO(q-1,1)$. Accordingly, we separate the spacetime indices ($\mu=0,\ldots,d-1$) into two subsets: orthogonal ($i=0,\ldots,q-1$) and parallel ($a=q,\ldots,d-1$) to the defect. Our main focus is the two-point function of identical scalar primaries in the spectrum of the ambient CFT. The correlator is a function of two cross-ratios as has been discussed previously in sec. 2.3.3. Let us choose $x_{12}^a=0$, and also choose the two bulk primaries to be on a plane in the transverse (x,t) space which contains the origin. The geometry is shown in fig. 3.2. The two standard cross-ratios can be traded for the lightcone coordinates $x_2=(z,\bar{z}):=(x+t,x-t)$ of the second insertion and we can express the two-point function as,

$$\langle\phi(x_1)\phi(x_2)\rangle=\frac{\mathcal{A}(z,\bar{z})}{(|(x_1)_\perp||(x_2)_\perp|)^{\Delta_\phi}}=\frac{\mathcal{A}(z,\bar{z})}{(z\bar{z})^{\Delta_\phi/2}}\,. \tag{3.6}$$

Some convenient features of the $(z,\bar{z})$ coordinates in the defect CFT context are discussed in [65]. The lightcone limits we shall be interested in involve bringing x_2 close to $\bar{z}=1$ (wherein the operator at x_2 is lightlike separated from the one at x_1), and also close to $z=0$ (which makes the operator at x_2 lightlike separated from the defect). We shall see that non-trivial constraints on the spectrum of the defect CFT can be imposed by examining the crossing equation in these limiting kinematics.

3.3 Lightcone bootstrap

As discussed in sec. 2.5.1 and sec. 2.6.1, the two-point function in the bulk CFT can be expanded in conformal blocks either in the bulk or in the defect channel. Crossing symmetry is the statement of equality of these two expansions:

$$\mathcal{A}(z,\bar{z}) = \left(\frac{(1-z)(1-\bar{z})}{(z\bar{z})^{1/2}}\right)^{-\Delta_\phi} \sum_{O} \lambda_{\phi\phi O}\, a_O\, g_{\Delta,l}(z,\bar{z}) = \sum_{\widehat{O}} (b_{\phi\widehat{O}})^2\, \widehat{g}_{\widehat{\tau},s}(z,\bar{z})\,. \tag{3.7}$$

The first sum runs over the bulk spectrum, and $g_{\Delta,l}(z,\bar{z})$ is the bulk conformal block for the exchange of a primary of quantum numbers (Δ, l). The prefactor is chosen so that $g_{0,0}(z,\bar{z}) = 1$. The OPE data entering the bulk channel is the product of a three-point function coefficient ($\lambda_{\phi\phi O}$) and the coefficient of the one-point function of the exchanged bulk operator (a_O). The second sum in eq. (3.7) runs over defect primaries. The latter do not carry $SO(p)$ spin when the external operators are scalars as the two-point function of a scalar bulk primary and a spinning defect primary is zero. The defect operators can however carry a charge s under the transverse $SO(q-1,1)$. The conformal blocks $\widehat{g}_{\widehat{\tau},s}(z,\bar{z})$ are then labeled by the transverse spin s and the transverse twist $\widehat{\tau}$. $b_{\phi\widehat{O}}$ is the coefficient[2] of the two-point function $\langle\phi\widehat{O}\rangle$.

The block in the defect channel is known exactly [64] and has been presented previously in sec. 2.5.1. We shall use the expression for the conformal block in eq. (2.93). We shall now label the blocks with transverse twist $\widehat{\tau}$ and transverse spin s and use $z,\bar{z}$ variables.

$$\widehat{g}_{\widehat{\tau},s}(z,\bar{z}) = z^{\widehat{\tau}/2}\bar{z}^{\widehat{\tau}/2+s}{}_2F_1\left(-s,\frac{q}{2}-1,2-\frac{q}{2}-s,\frac{z}{\bar{z}}\right){}_2F_1\left(\widehat{\tau}+s,\frac{p}{2},\widehat{\tau}+s+1-\frac{p}{2},z\bar{z}\right)\,. \tag{3.8}$$

When q is even, an order of limits ambiguity arises in the definition of the hypergeometric function, one must first take s to be integer, and then q to be even. This prescription is henceforth assumed - see appendix. A for details.

The bulk-channel conformal blocks are not known in closed form for generic dimension and codimension apart from $q = 2$ and $d = 4,6$ [64], and $q = 3$ and $d = 4$ [71]. Presently, we shall be primarily interested in the lightcone limit $\bar{z} \to 1$ and in this limit the (collinear) bulk blocks are given, for any d and q, by

$$\begin{aligned} g_{\Delta,l}(z,\bar{z}) &= (1-\bar{z})^{\frac{\Delta-l}{2}} \\ &\left(2^{-l}\left(\frac{(1-z)}{z^{1/2}}\right)^{\frac{\Delta+l}{2}} {}_2F_1\left(\frac{\Delta+l}{4},\frac{\Delta+l}{4},\frac{\Delta+l+1}{2},-\frac{(z-1)^2}{4z}\right) + O((1-\bar{z}))\right)\,. \end{aligned} \tag{3.9}$$

Let us now ask if there can be a solution to the crossing eq. (3.7) with a finite number of blocks in either the bulk or defect decompositions. On the bulk side the answer is clearly yes: the trivial defect, *i.e.*, the two-point function in a CFT without a defect has a single block contributing to the bulk channel, that of the identity, and this obviously satisfies the

[2] We use a few slightly different conventions for the subscripts labeling $b_{\phi\widehat{O}}$ throughout the chapter depending on the context and expect that the notation is self-explanatory.

requiremnt of crossing symmetry. Whether it is possible to have a solution with finitely many non-trivial bulk primaries, on top of the identity, is a question that we do not address here. On the defect side, we can prove that the crossing eq. (3.7) cannot be satisfied by finitely many defect primary operators if $q > 1$. This is because the defect block in eq. (3.8) has an unphysical singularity when $z\bar{z} = 1$, for any value of $\frac{z}{\bar{z}}$, which is not consistent with the singularity structure of the two-point function in a Euclidean configuration. Let us discuss this explicitly.

For $p > 1$ the behavior of the defect block (3.8) for $z\bar{z} \to 1$ is

$$\widehat{g}_{\widehat{\tau},s}(z,\bar{z}) \overset{z\bar{z}\to 1}{\sim} \frac{\Gamma\left(\frac{p-1}{2}\right)\Gamma\left(\widehat{\Delta}-\frac{p-2}{2}\right)}{2^{2-p}\sqrt{\pi}\Gamma(\widehat{\Delta})}\left(\frac{\bar{z}}{z}\right)^{s/2} {}_2F_1\left(-s,\frac{q}{2}-1,2-\frac{q}{2}-s,\frac{z}{\bar{z}}\right)(1-z\bar{z})^{-p+1}\,, \quad p>1\,. \tag{3.10}$$

For $p = 1$ the singularity is a logarithm, and the following argument is unchanged. The case where $\widehat{\Delta} = \frac{p-2}{2}$ looks different in (3.10), but in this case too the argument goes through. It can be checked that all the coefficients of the hypergeometric function in (3.10), in an expansion around $z = 0$ which is a polynomial of degree s, are positive. Through the scalar unitarity bound (see eq. (2.98)), the sign of the prefactor in (3.10), specifically of the factor $\Gamma\left(\widehat{\Delta}-\frac{p-2}{2}\right)$ is also fixed independently of the spectrum. These arguments, alongwith the positivity of the defect channel OPE $(b_{\phi\widehat{O}})^2$ in eq. (3.7), implies that the singularity at $z\bar{z} = 1$ in the blocks cannot be canceled if there are only a finite number of blocks contributing to the defect channel expansion.

In the case of a codimension one defect the same singularity is instead potentially physical, and the exponent matches the exchange of the identity in the bulk channel when the theory is free. This allows for the existence of solutions to the boundary crossing equation with finitely many blocks [44, 63].

From the literature [16, 27, 28] on lightcone bootstrap in CFT discussed in sec. 3.1, we learn the lesson that analytic information can still be extracted from a crossing equation that contains infinitely many terms, by focusing on a limit that drastically simplifies one of the channels. In the case of the two-point function with a defect, there are two such simplifying limits. In the $z \to 0$ limit $\phi(x_2)$ is light-like separated from the defect, and the defect channel OPE is dominated by the operators with the smallest $\widehat{\tau} = \widehat{\Delta} - s$, as can be seen explicitly in eq. (3.8). The other lightcone limit is $(1-\bar{z}) \to 0$ wherein $\phi(x_2)$ is light-like separated from $\phi(x_1)$. This is the limit that we will focus on in this section.

Now we come to the crux of the entire story. When $(1-\bar{z}) \to 0$ the identity contribution dominates the bulk channel expansion, as is clear from (3.9). On the other hand, there are infinitely many operators contributing to the defect channel expansion. But the indentity contribution to the bulk channel is theory independent. This leads us to hope that some universal (analytic) statement can be made about the defect spectrum by focussing on this lightcone limit where the behavior in the bulk channel is theory independent. It remains to be shown explicitly that there exists a specific sector of the defect spectrum that is mostly sensitive to the identity in the crossed channel and that a perturbation theory around this limiting behavior can be set up that captures the theory dependent deviations from the universality. We postpone this discussion to subsection 3.3.3. For now, let us try to obtain the theory independent statement about the defect spectrum that we expect to discover from the strict lightcone limit $(1-\bar{z}) \to 0$.

Since we are looking for a theory independent statement, we could choose any particular theory that is easily accesssible and obtain the desired results. And the simplest CFT with a defect is a CFT with no defect. Indeed, when there is no defect the identity is the only exchanged primary in the bulk channel and the transverse derivative operators in the defect spectrum can be expressed in terms of bulk operators as shown in sec. 3.3.1.

3.3.1 The trivial defect

Let us consider the two-point function of identical bulk scalar primaries in CFT (without defect).

$$\mathcal{A}(z,\bar{z}) = \langle \phi(x_1)\phi(x_2)\rangle \, (z\bar{z})^{\Delta_\phi/2} = \left(\frac{(z\bar{z})^{1/2}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} . \tag{3.11}$$

We can take any hyperplane to be the (trivial) defect and expand $\langle\phi(x_1)\phi(x_2)\rangle$ in a Taylor series of (two-point functions of) operators localized on this hyperplane. We can organize this expansion into blocks that are irreducible under the subgroup of the conformal group that fixes the hyperplane chosen as the defect and thus we have a valid defect channel expansion of the two-point function. The defect OPE of the primary ϕ of dimension Δ_ϕ is regular and only contains primaries of the kind

$$(\partial_i)^n(\partial_j\partial^j)^m\phi\,, \qquad \begin{cases} \widehat{\Delta} = \Delta_\phi + n + 2m\,, \\ s = n\,. \end{cases} \tag{3.12}$$

When localized on the hyperplane, these are primaries of the defect CFT as the derivatives of ϕ in (3.12) are transverse to the defect. The defect channel expansion of $\mathcal{A}(z,\bar{z})$ is in blocks corresponding to these defect primaries. These are the transverse derivative operators in this theory of a trivial defect. The non-trivial statement we wish to make is that there exist towers of operators with their transverse twists approaching this limiting value for $s \to \infty$ in the spectrum of any defect CFT. We shall use this schematic form (3.12) for the transverse derivative operators although it may not make sense to denote them as such in the context of every theory (for example when s takes half-integer values as it does in the 3D Ising twist defect discussed in sec. 3.6).

The defect channel OPE coefficients are just the coefficients in the expansion of $\mathcal{A}(z,\bar{z})$ discussed above and these can be expressed in closed form.

$$b^2_{s,m} = \frac{\Gamma\left(\frac{q}{2}+s\right)\Gamma(2m+s+\Delta_\phi)\Gamma\left(m-\frac{d}{2}+\Delta_\phi+1\right)\Gamma\left(m-\frac{p}{2}+s+\Delta_\phi\right)}{\Gamma(\Delta_\phi)\Gamma(m+1)\Gamma(s+1)\Gamma\left(m+\frac{q}{2}+s\right)\Gamma\left(-\frac{d}{2}+\Delta_\phi+1\right)\Gamma\left(2m-\frac{p}{2}+s+\Delta_\phi\right)} . \tag{3.13}$$

We shall now argue that, at large s, the spectrum of any defect does contain a sector close to the trivial defect, in the same sense as ordinary CFT spectra are close to generalized free theory in the large spin limit. Our strategy in the rest of the section is analogous to the one presented in [27, 28]. Certain weak points in the traditional arguments for the lightcone bootstrap have been pointed out in [204]. It may be expected that these issues are easier to tackle here with respect to the analogous ones in the lightcone bootstrap of the four-point function [27, 28], due to the simplicity of the defect channel blocks. However, we do not have a completely rigorous presentation of the lightcone bootstrap yet and only point out the issues that need to be addressed. Nevertheless, we shall define a calculable perturbative series, whose predictions we shall test in some examples in sec. 3.6. Furthermore, in subsection 3.4.3 we shall come back to the question with a more powerful tool in our hands, which will allow us to rigorously prove the results that follow.

3.3.2 The defect spectrum at large transverse spin: zeroth order

In what follows, we would like to analyze the crossing equation (3.7) in the bulk light-cone limit, and more specifically in the following region:

$$1-\bar{z} \ll z < 1\,. \tag{3.14}$$

In this regime, the contribution of the higher twist bulk operators is suppressed with respect to the identity as evident from the lightcone approximation of the bulk block in eq. (3.9). The defect OPE still converges and hence we can rewrite the crossing equation (3.7) as follows:

$$1 = \lim_{\bar{z}\to 1}\left(\frac{(1-z)(1-\bar{z})}{\sqrt{z\bar{z}}}\right)^{\Delta_\phi}\sum_{\widehat{\tau},s}(b_{\phi\widehat{\mathcal{O}}})^2\,\widehat{g}_{\widehat{\tau},s}(z,\bar{z})\,. \tag{3.15}$$

The conformal blocks of (a subsector of) the defect primaries need to cancel the $(1-\bar{z})$ dependence in the prefactor. First we ask if we are allowed to exchange the order of the limit and the sum in eq. (3.15). However, each conformal block is analytic at $\bar{z}=1$, as long as $z<1$, therefore $\lim_{\bar{z}\to 1}(1-\bar{z})^{\Delta_\phi}\widehat{g}_{\widehat{\tau},s}=0$ for every operator in the spectrum. We are led to conclude that we must not be allowed to exchange the order of (at least one of) the sums and the limit in eq. (3.15) and hence (at least one of) the sums does not converge uniformly at $\bar{z}=1$. Therefore let us look for the region which is responsible for the singularity and consider the two sums over $\widehat{\tau}$ and s carefully.

At large and positive $\widehat{\tau}$, for fixed s, the blocks are suppressed by $z^{\widehat{\tau}/2}$ for every $z<1$ - see eq. (3.8). Since $\widehat{\tau}$ is not subject to a unitarity bound, one may worry about the $\widehat{\tau}\to-\infty$ as well. This limit corresponds to the large s limit at fixed $\widehat{\Delta}$, and it is relevant only if the spectrum contains infinitely many operators of growing transverse spin in a finite interval in $\widehat{\Delta}$. This situation appears to be very peculiar, and we generically expect the sum over s to be bounded at fixed value of the scaling dimension. In the heuristic spirit of the discussion, we will make this assumption here and move ahead.

However, precisely in the limit $\bar{z}\to 1$ the sum over transverse spins, at fixed $\widehat{\tau}$, ceases to be suppressed. Therefore, in studying this last region, we shall replace the blocks with their large s asymptotics:

$$\widehat{g}_{\widehat{\tau},s}(z,\bar{z}) \underset{\widehat{\tau}\text{ fixed}}{\overset{s\to\infty}{\sim}} (z\bar{z})^{\widehat{\tau}/2}\bar{z}^{s}\left(\frac{\bar{z}}{(\bar{z}-z)}\right)^{\frac{q-2}{2}}(1-z\bar{z})^{-\frac{p}{2}}\left(1+\mathcal{O}\left(\frac{1}{s}\right)\right)\,. \tag{3.16}$$

This approximation is obtained in the region $0<z<\bar{z}<1$. Notice that the blocks are nicely factorized in the pairs $(z\bar{z},\widehat{\tau})$ and $(\bar{z},s)$ in this limit.

Now we come to the important step where we show the existence of the transverse derivative operators in this generic defect CFT spectrum. Let us plug in the asymptotics (3.16) in eq. (3.15), and analyze the equation order by order in z. In order that the left hand side matches the right hand side in eq. (3.15), we must have powers of z from the sum over the defect blocks that cancel with the powers of z from the prefactor on the right hand side. Thus we are led to conclude that the following accumulation points exist in the defect spectrum for asymptotically large s:

$$\widehat{\tau} = \Delta_\phi + 2m + \mathcal{O}(s^{-\alpha}) \ , \quad m \in \mathbb{N} \, , \quad s \to \infty \, , \tag{3.17}$$

with m being a non-negative integer, and α being a real positive number that is yet to be determined.

The presence of a finite s correction must be allowed, because we only treated the crossing equation (3.15) in the strict infinite s limit. The operators (3.17) are obviously the transverse derivatives described in eq. (3.1).

To be precise, we also want to show that there are infinitely many operators (differing in s) whose transverse twists approach (3.17) for asymptotically large s. To this end let us rewrite the crossing equation once again,

$$\left(\frac{\sqrt{z}}{1-z}\right)^{\Delta_\phi} (1-\bar{z})^{-\Delta_\phi} = \lim_{\bar{z}\to 1} \bar{z}^{-\frac{\Delta_\phi}{2}} \sum_{\widehat{\tau}} (z\bar{z})^{\frac{\widehat{\tau}}{2}} \left(\frac{\bar{z}}{\bar{z}-z}\right)^{\frac{q-2}{2}} (1-z\bar{z})^{-\frac{p}{2}} \sum_s b^2_{\widehat{\tau},s} \bar{z}^s \, . \tag{3.18}$$

For each power of z on the left hand side, we have an associated $(1-\bar{z})^{-\Delta_\phi}$ singularity. As discussed earlier, this singularity can be reproduced from the right hand side with only an infinite sum over operators (with different values of s). Thus to recapitualte what we have shown so far: every defect CFT with codimension $q \geq 2$ has accumulation points in the transverse twist spectrum at $\widehat{\tau} = \Delta_\phi + 2m$ for each $m \in \mathbb{N}$. From the discussion above, the role of the (double) lightcone limits in simplifying the crossing equation is also established. While the $(1-\bar{z}) \to 0$ limit zooms onto the contribution of the large spin sector of the defect spectrum where the transverse derivative operators dominate, $z \to 0$ limit controls how many towers (labelled by m) of transverse derivative operators we focus on.

With the spectrum (3.17) now available, we wish to determine the asymptotic behavior of the defect channel OPE coefficients for large s. The natural guess is that the OPE coefficients should approach the large s behavior of the corresponding ones in the trivial defect theory. Employing the large s asymptotic behavior of (3.13) in eq. (3.15) with the spectrum (3.17), we can verify that the crossing equation is indeed satisfied. Therefore,

$$b^2_{s,m} = s^{\Delta_\phi - 1}\left(\frac{1}{\Gamma(\Delta_\phi)}\binom{m-\frac{d}{2}+\Delta_\phi}{m} + \mathcal{O}(s^{-\beta})\right) \ , \quad s \to \infty \ , \tag{3.19}$$

for some positive β.

Let us look in detail at the way the identity in eq. (3.15) is reproduced at leading order in $1-\bar{z}$. This highlights the relation between the large s and small $1-\bar{z}$ limits [27, 28]. Let us write the right hand side of the crossing equation (3.7) only including the transverse derivatives and using the large s asymptotics of the blocks eq. (3.16) and of the OPE coefficients eq. (3.19). Since we are only interested in the contributions from the large spin part of the spectrum, we also replace the sum over spins by an integral and introduce a lower cutoff to the integral at some spin Λ:

$$\left(\frac{\bar{z}}{(\bar{z}-z)}\right)^{\frac{q-2}{2}}(1-z\bar{z})^{-\frac{p}{2}}\sum_{m=0}^{\infty}\binom{m-\frac{d}{2}+\Delta_\phi}{m}(z\bar{z})^{\Delta_\phi/2+m}\left(\frac{1}{\Gamma(\Delta_\phi)}\int_\Lambda^\infty ds s^{\Delta_\phi-1}\bar{z}^s\right)$$
$$=\left(\frac{\sqrt{z\bar{z}}}{1-z}\right)^{\Delta_\phi}\frac{\Gamma(\Delta_\phi,-\Lambda\log\bar{z})}{\Gamma(\Delta_\phi)}\frac{1}{(-\log\bar{z})^{\Delta_\phi}}\,. \tag{3.20}$$

In the $\bar{z}\to 1$ limit, the result matches the bulk identity, for any finite Λ, however large. This confirms that only asymptotically large values of s matter. In fact, Λ could even be sent to infinity, as long as the growth is slower than $1/(1-\bar{z})$. This signals which range of spin is important in reproducing the bulk OPE limit, and cannot be excluded from the integral:

$$s\sim\frac{1}{1-\bar{z}}\,. \tag{3.21}$$

An alternative way to understand this fact is through a saddle point approximation of the simple integral in eq. (3.20), as in [28], which is accurate for large Δ_ϕ. The relation (3.21) can be contrasted with the one relevant to double twist operators, that is $l\sim 1/(1-\bar{z})^{1/2}$ [27, 28]. The different behavior here is responsible for the different finite spin exponent α - see eq. (3.17) below - of the transverse derivatives with respect to the one of double twists.

Let us emphasize why the derivation above is not entirely rigorous and how we could rectify that situation – see appendix F in [204] for a more detailed discussion. In many of the steps above, we have implicitly assumed that since the sum over $\widehat{\tau}$ on the defect channel converges, we are allowed to exchange the orders of the limit $\bar{z}\to 1$ and the sum over $\widehat{\tau}$ once we have evaluated the infinite sum over s. However the convergence of $\widehat{\tau}$ sum doesn't entirely justify the exchange. To do that, one should also prove that the following limit exists at fixed $\widehat{\tau}$:

$$\rho(\widehat{\tau})=\lim_{\bar{z}\to1}\sum_{s=0}^{\infty}(b_{\phi\widehat{O}})^2\,(1-\bar{z})^{\Delta_\phi}\bar{z}^s\,. \tag{3.22}$$

One should then plug $\rho(\widehat{\tau})$ in eq. (3.15) and rewrite the crossing eq. (3.15) with a conformal block density in the transverse twist space. This is a more rigorous approach to the simple expansion in orders of z around 0 and will establish the existence of the transverse derivative operators. Up to the issue of bounding the spectrum at negative $\widehat{\tau}$, this step can be done precisely as in [27]. The rest is then equivalent to the previous discussion: since the trivial defect in particular solves eq. (3.15), we obtain $\rho(\widehat{\tau})$ by plugging the OPE coefficient of the trivial defect in eq. (3.22). At this point, in turn, the Hardy-Littlewood tauberian theorem [204] can be used to deduce from eq. (3.22) the asymptotics (3.19). Finally, we stress that eq. (3.19) establishes an averaged property of the spectrum at large spin, while we have no control on the OPE coefficient of single defect primaries.

3.3.3 The defect spectrum at large transverse spin: higher orders

Finite spin corrections to eqs. (3.17) and (3.19) can be computed by taking into account subleading contributions to the bulk channel expansion in eq. (3.7) in the $\bar{z}\to 1$ limit. These subleading contributions can come from the identity contribution itself and also from bulk blocks with low values of twist that contribute at a subleading order with respect to the identity. In order to make a consistent truncation of the bulk channel expansion in eq. (3.7) at some order in $1-\bar{z}$, one needs to have information on the bulk spectrum. We wish to keep our discussion completely general and therefore we shall only strive to calculate finite spin corrections to eqs. (3.17) and (3.19) corresponding to the contribution of a single block (with any dimension Δ and spin l) to the bulk channel.

First let us consider finite spin corrections to the OPE data of the transverse derivative sector resulting from subleading contributions from the identity block itself. It turns out that a rather trivial series of corrections is required to match higher orders in $1-\bar{z}$ coming from the bulk identity block. No finite spin corrections to the values of transverse twist are required to satisfy crossing symmetry. Only the OPE coefficients are affected, and all of the corrections are trivially obtained by expanding eq. (3.13) at large s.

More interestingly, new bulk primaries start contributing at some order in $1-\bar{z}$, according to their twist. As mentioned before, we restrict the analysis to the corrections due to a single bulk block and this discussion also equips us to calculate corrections due to any finite number of bulk blocks. In the context of weakly coupled CFTs, we might encounter a scenario wherein infinitely many primaries exist with (nearly) degenerate values of twist. This problem relevant to weakly interacting CFTs has been attended to in [29, 198] for the four-point function without defects. We leave this analysis for CFTs with defects for future work.

We assume that the leading contribution to the bulk channel after the identity comes from a single bulk block with minimal twist τ and dimension Δ (spin l). The crossing eq. (3.7) can now be written as,

$$\left(\frac{(z\bar{z})^{1/2}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi}\left(1+\lambda_{\phi\phi O}\,a_O\,g_{\Delta,l}(z,\bar{z})\right)\sim\sum_{\substack{s\text{ large}\\ \widehat{\Delta}-s=\widehat{\tau}(s)}} b^2_{s,\widehat{\tau}(s)}\,\widehat{g}_{\widehat{\tau}(s),s}(z,\bar{z})\,. \tag{3.23}$$

We shall solve the crossing eq. (3.23) to subleading order in $(1-\bar{z})$ and expect that this will give us corrections to the twist trajectories $\widehat{\tau}(s)$ in (3.17) and OPE coefficients in (3.19). Note that so far it is only a guess that by incorporating $\frac{1}{s}$ corrections to the leading behavior in equations (3.17) and (3.19) we should be able to solve the crossing eq. (3.23). Now we shall show explicitly that we can account for the leading twist operator O in the bulk channel by modifying the trajectory $\widehat{\tau}(s)$ of the transverse derivative operators and their OPE coefficients. We shall restrict the discussion to the leading transverse twist trajectory corresponding to $m=0$ (in (3.17)) to avoid cluttering the equations unnecessarily. This corresponds to focussing on the strict $z\to 0$ limit of the crossing eq. (3.23).

The contribution from the minimal twist (collinear) bulk block in eq. (3.9) in the strict $z\to 0$ limit is given by,

$$g_{\Delta,l}(z,\bar{z})\ \overset{z\to 0}{\sim}\ -2^{\Delta-1}\frac{\Gamma\left(\frac{1}{2}+\frac{\Delta+l}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\Delta+l}{2}\right)}(1-\bar{z})^{\frac{\Delta-l}{2}}\left(2\left(\gamma_E+\psi\left(\frac{\Delta+l}{2}\right)\right)+\log z\right)\,. \tag{3.24}$$

Note that if we plug this block back in eq. (3.23), the contribution is singular as $\bar{z}\to 1$ only if $\Delta_\phi-\frac{\Delta-J}{2}<0$, in which case the rest of the discussion follows directly. For now we shall assume this and we shall comment on the opposite scenario later. The left hand side of the crossing eq. (3.23) is then simply given by:

$$\left(\frac{(z\bar{z})^{1/2}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi}\left(1-\lambda_{\phi\phi O}\,a_O\,2^{\Delta-1}\frac{\Gamma\left(\frac{1}{2}+\frac{\Delta+l}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\Delta+l}{2}\right)}(1-\bar{z})^{\frac{\Delta-l}{2}}\right.$$
$$\left.\left(2\left(\gamma_E+\psi\left(\frac{\Delta+l}{2}\right)\right)+\log z\right)\right). \tag{3.25}$$

Let us assume the following parametrization for the leading transverse twist trajectory and the corresponding defect channel OPE coefficient:

$$\widehat{\tau}(s) = \Delta_\phi + \frac{c_{\text{min}}}{s^\alpha}\,, \tag{3.26}$$

$$b_s^2 = \frac{\Gamma\left(\Delta_\phi+s\right)}{\Gamma(s+1)\Gamma(\Delta_\phi)}\left(1+\frac{b_{\text{min}}}{s^\beta}\right). \tag{3.27}$$

We expect the logarithmic term in the block in eq. (3.24) to be responsible for the anomalous dimension in eq. (3.26) and the log-independent terms to be responsible to the corrections to the OPE coefficient as in eq. (3.27). Note that a consistent truncation (in orders of $1/s$) of b_s^2 in eq. (3.27) depends on the precise values of α and β. The factor $\frac{\Gamma\left(\Delta_\phi+s\right)}{\Gamma(s+1)\Gamma(\Delta_\phi)}$ in eq. (3.27) is the contribution to the OPE coefficient from the identity exchange. Since we are only interested in the finite spin corrections corresponding to the minimal twist bulk block exchange (over the identity), we shall choose to extract the entire contribution from the identity exchange as in eq. (3.27).

With the parametrization in eq. (3.26) the defect channel block (at large s) in eq. (3.16) can now be written as:

$$\begin{aligned}\widehat{g}_{\widehat{\tau},s}(z,\bar{z}) &\sim (z\bar{z})^{\frac{\Delta_\phi}{2}+\frac{c_{\text{min}}}{2s^\alpha}}\,\bar{z}^s\left(\frac{\bar{z}}{(\bar{z}-z)}\right)^{\frac{q-2}{2}}(1-z\bar{z})^{-\frac{p}{2}}\,,\\ &\overset{\bar{z}\to 1}{\approx} (z\bar{z})^{\frac{\Delta_\phi}{2}}\,\bar{z}^s\left(\frac{\bar{z}}{(\bar{z}-z)}\right)^{\frac{q-2}{2}}(1-z\bar{z})^{-\frac{p}{2}}\left(1+\frac{c_{\text{min}}}{2s^\alpha}\log z\bar{z}\right).\end{aligned} \tag{3.28}$$

With the OPE coefficients in eq. (3.27), we get the right hand side of the crossing eq. (3.23) as follows:

$$\sum_{\substack{s\text{ large}\\ \widehat{\Delta}-s=\widehat{\tau}(s)}} b^2_{s,\widehat{\tau}(s)}\,\widehat{g}_{\widehat{\tau}(s),s}(z,\bar{z}) \approx \sum_{\substack{s\text{ large}\\ \widehat{\Delta}-s=\widehat{\tau}(s)}}\frac{\Gamma\left(\Delta_\phi+s\right)}{\Gamma(s+1)\Gamma(\Delta_\phi)}(z\bar{z})^{\frac{\Delta_\phi}{2}}\,\bar{z}^s\left(\frac{\bar{z}}{(\bar{z}-z)}\right)^{\frac{q-2}{2}}$$
$$(1-z\bar{z})^{-\frac{p}{2}}\left(1+\frac{b_{\text{min}}}{s^\beta}+\frac{c_{\text{min}}}{2s^\alpha}\log z\bar{z}\right), \tag{3.29}$$

In the sum $\left(1+\frac{b_{\text{min}}}{s^\beta}+\frac{c_{\text{min}}}{2s^\alpha}\right)$ above, the first term accounts for the identity contribution to the bulk channel expansion, the second and third terms account for the contribution of the minimal twist block.

Now we wish to match the terms from in (3.25) and (3.29), in the limits $z \to 0$ and $(1-\bar{z}) \to 0$. The identity contribution has been matched already. Matching the logarithmic terms gives us the anomalous dimension with:

$$\alpha = \frac{\tau}{2} = \frac{\Delta - l}{2}\,, \tag{3.30}$$

and

$$c_{\text{min}} = -\lambda_{\phi\phi O}\, a_O 2^{\Delta} \frac{\Gamma(\Delta_\phi)}{\Gamma\left(\Delta_\phi - \frac{\tau}{2}\right)} \frac{\Gamma\left(\frac{1}{2} + \frac{\Delta+l}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\Delta+l}{2}\right)}\,. \tag{3.31}$$

Similarly, the correction to the OPE coefficient (3.19) is given by:

$$\beta = \frac{\tau}{2} = \frac{\Delta - l}{2}\,, \tag{3.32}$$

and

$$b_{\text{min}} = -c_{\phi\phi O}\, a_O\, 2^{\Delta} \frac{\Gamma(\Delta_\phi)}{\Gamma\left(\Delta_\phi - \frac{\tau}{2}\right)} \frac{\Gamma\left(\frac{1}{2} + \frac{\Delta+l}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\Delta+l}{2}\right)} \left(\gamma_E + \psi\left(\frac{\Delta+l}{2}\right)\right)\,. \tag{3.33}$$

Thus we have explicitly verified, for the leading transverse twist trajectory, that we can indeed account for subleading corrections in $1-\bar{z}$ to the bulk channel expansion by incorporating finite spin corrections of the form (3.26) and (3.27) to the transverse twist trajectory and the OPE coefficients respectively and have also calculated these corrections in closed form.

We should now mention a shortcoming in the derivation above. Although the power-law form of the subleading corrections to eqs. (3.17-3.19) was sufficient to account for subleading corrections in $1-\bar{z}$ to the bulk expansion, we are not guaranteed that this is the unique solution to the problem. So far, we do not know how to establish this rigorously (without resorting to the OPE inversion formula in sec. 3.4.2).

Let us pause to comment on the non-singular case $\Delta_\phi - \frac{\Delta-l}{2} > 0$. Following [31], one can act with the defect Casimir operator $\mathcal{C}_{\text{def}}$, written down in [64], on both sides of the crossing equation. On the bulk side we find

$$\mathcal{C}_{\text{def}}\left[(1-\bar{z})^{\delta} f(z)\right] = -2\delta(\delta-1)(1-\bar{z})^{\delta-2} f(z) + \mathcal{O}(1-\bar{z})^{\delta-1}\,, \tag{3.34}$$

and so the leading behavior of (3.24) can be made singular by repeatedly acting with the defect Casimir, provided $\frac{\Delta-l}{2} - \Delta_\phi$ is not a positive integer. For generic Δ, Δ_ϕ the contribution of a bulk primary is thus Casimir-singular in the sense of [33]. On the defect side, acting with $\mathcal{C}_{\text{def}}$ introduces the eigenvalue for the corresponding defect block, which grows as s^2 for large s and thus enhances the large s behavior. Therefore, the results (3.31-3.33) are valid also if $\Delta_\phi - \frac{\Delta-l}{2} > 0$ and non integer.

Here we presented the result for the first subleading corrections to the leading transverse twist trajectory and the corresponding OPE coefficients, but similar corrections to (3.17) and (3.19) for $m \neq 0$ are straightforward to obtain. The large s expansion of anomalous dimensions and OPE coefficients can be set up systematically to obtain the contribution of a collinear primary and all its descendants, as done in [30] for the four-point function case. The only requirement is the knowledge of the subleading contributions to (3.9). However, we shall not pursue that direction. In section 3.4, we will obtain an inversion formula for the defect OPE, analogous to the one found in [74] for the four-point function, which allows to resum the lightcone expansion thus also establishing the results from the lightcone bootstrap rigorously.

3.4 Inversion of the defect channel OPE

In this section we describe a general way to extract the defect spectrum given a two-point function of bulk primaries. The quantum numbers $(\widehat{\tau},\, s)$ and the defect OPE coefficient $(b_{\phi\widehat{\mathcal{O}}})^2$ are extracted by an integral transform of the two-point function, which is analytic in the transverse spin s. This is the defect CFT analog of the inversion formula from [74] (discussed briefly in sec. 3.1), which applies to four-point functions in theories without defects. Most of the features of the integral transform that we shall discuss here, and its derivation, are analogous to that of [74].

This OPE inversion formula obtained in this section allows to resum the large s results of section 3.3, and thus obtain the scaling dimension of defect operators with finite transverse spin and corresponding defect channel OPE coefficients. It also bypasses the need for some of the assumptions required by the lightcone analysis that were discussed in sec. 3.3 thus putting the results on a rigorous footing. The validity of the integral transform, similarly to that of [74], depends on the growth of the correlator in a certain region. Contrary to [74], though, the behavior of the correlator in this region is not controlled by an OPE limit, and we cannot place general bounds on its growth. We shall further comment on this issue in subsection 3.4.2.

In the rest of this section we derive the inversion formula for the defect OPE following in the footsteps of [74]. For this purpose we start by obtaining a Euclidean inversion formula, which simply follows from orthogonality of partial waves (see for example [205]). In the context of the toy example that we discussed in sec. 3.1, this Euclidean formula would be analogous to eq. (3.3). While this Euclidean formula is not analytic in the transverse spin s, it can be manipulated into a Lorentzian formula that is. This is analogous to the contour deformation arguments taking us from eq. (3.3) to eq. (3.4). An alternate derivation of the Lorentzian inversion formula relevant to the four-point function in theories without defects was presented recently in [75], but we shall not discuss such an alternate derivation for the present case.

3.4.1 The Euclidean inversion formula

Let us start by obtaining an Euclidean inversion formula for the bulk-defect OPE. Recall that in our configuration (see sec. 3.2) the two operators lie on a plane orthogonal to the defect. The defect intersects the plane at the origin, with one external operator placed at $x_1 = (1,1)$ and the other at $x_2 = (z,\bar{z})$, see fig. 3.2. We introduce the following radial coordinates for the position of the second operator

$$z = rw\,, \qquad \bar{z} = \frac{r}{w}\,, \qquad \eta = \frac{1}{2}\left(w + \frac{1}{w}\right)\,. \tag{3.35}$$

In Euclidean signature $\bar{z} = z^*$ and so w is a phase. Since defect blocks (3.8) factorize, we can write,

$$\widehat{g}_{\widehat{\tau},s}(z,\bar{z}) = \widehat{f}_s(\eta)\widehat{g}_{\widehat{\Delta}}(r)\,, \tag{3.36}$$

where $\widehat{\tau} = \widehat{\Delta} - s$.

To obtain the Euclidean inversion formula, the strategy is to relate the bulk-defect OPE to a completeness relation which can be easily inverted using the orthonormality of the basis involved in this completeness relation. Naturally, we can expect Sturm-Liouville theory to play a crucial role in this as the (normalized) eigenfunctions of a Sturm-Liouville operator form an orthonormal basis for an appropriate Hilbert space. Now we shall look into the radial $\widehat{g}_{\widehat{\Delta}}(r)$ and angular $\widehat{f}_s(\eta)$ parts separately and obtain the relevant orthogonality relations in each case.

Radial factor of defect blocks

Let us start by considering the parallel factor of the block. As discussed in sec. 2.5.1, the parallel and angular parts of the defect channel conformal blocks satisfy separate Casimir eigenvalue equations. Similarly to [205], we re-write the Casimir equation that $\widehat{g}_{\widehat{\Delta}}(r)$ satisfies (see [64]) in the form of a Sturm-Liouville problem:

$$\mathcal{D}_{\|}\widehat{g}(r) = \widehat{\Delta}(\widehat{\Delta}-p)\widehat{g}(r)\,, \qquad \text{with} \qquad \mathcal{D}_{\|}\widehat{g}(r) = \frac{r^{p+1}}{(1-r^2)^p}\frac{d}{dr}\left(r^{1-p}(1-r^2)^p\,\frac{d\widehat{g}(r)}{dr}\right). \tag{3.37}$$

The operator $\mathcal{D}_{\|}$ defined in (3.37) is self-adjoint on the Hilbert space $L^2[[0,1]\,,\mu_p(r)dr]$ where the measure $\mu_p(r)$ is given by:

$$\mu_p(r) = \frac{(1-r^2)^p}{r^{p+1}}\,. \tag{3.38}$$

In this Hilbert space, the operator $\mathcal{D}_{\|}$ acts on functions that are continuously differentiable on $[0,1]$ and sufficiently well-behaved near the boundaries of the interval i.e. at $r=0$ and $r=1$ such that they are square-integrable with respect to the measure $\mu_p(r)$. We wish to find the spectrum of eigenvalues and the eigenfunctions of this operator which will give us the desired complete basis for this Hilbert space. We can expect that the spectrum of eigenvalues of the operator $\mathcal{D}_{\|}$ will have a continous component as the coefficients $\frac{r^{p+1}}{(1-r^2)^p}$ and $r^{1-p}(1-r^2)^p$ are not regular at the the boundaries of $[0,1]$ in general thus making it a singular Sturm-Liouville problem.

Concretely, self-adjointness requires that the following boundary term vanishes

$$\begin{aligned}\int_0^1 dr\,\mu_p(r)\,\mathcal{D}_{\|}(\Psi(r))\tilde{\Psi}(r) - \int_0^1 dr\,\mu_p(r)\,\Psi(r)\mathcal{D}_{\|}(\tilde{\Psi}(r))\\ = \int_0^1 dr\,\frac{d}{dr}\left[\mu_p(r)r^2\left(\frac{d\Psi(r)}{dr}\tilde{\Psi}(r) - \Psi(r)\frac{d\tilde{\Psi}(r)}{dr}\right)\right].\end{aligned} \tag{3.39}$$

We can easily check from eq. (3.39) that for the functions $\Psi(r)$ to be square-integrable with respect to the measure (3.38) their behavior near $r=0$ and $r=1$ must be such that

$$\Psi(r) \underset{r\to 0}{\sim} r^{\frac{p}{2}+\epsilon}\,, \qquad \Psi(r) \underset{r\to 1}{\sim} (1-r)^{-\frac{p+1}{2}+\epsilon'}\,, \tag{3.40}$$

where ϵ, ϵ' are positive numbers. However, the parallel factor in the defect conformal blocks,

$$\widehat{g}_{\widehat{\Delta}}(r) = r^{\widehat{\Delta}}\,{}_2F_1\left(\widehat{\Delta},\frac{p}{2},\widehat{\Delta}+1-\frac{p}{2},r^2\right)\,, \tag{3.41}$$

which is an eigenfunction of $\mathcal{D}_{\|}$, grows as $(1-r)^{1-p}$ for $r\to 1$ (this growth is logarithmic in the $p=1$ case). Therefore, unless $p=1,2$ their square is not integrable against the measure (3.38), and for no value of p does the boundary term in (3.39) vanish.

From this, we realize that although the parallel factor of the conformal block is indeed a solution of the appropriate Casimir eigenvalue equation, it is not a solution to the Sturm-Liouville problem with the given boundary conditions and thus the parallel factor of the blocks does not provide us with an orthonormal basis for the L^2 space on which this operator is self-adjoint.

Following [74, 205] we consider a linear combination of $\widehat{f}_{\widehat{\Delta}}$ that is still an eigenfunction of $\mathcal{D}_{\|}$, with eigenvalue $\widehat{\Delta}(\widehat{\Delta}-p)$, but is regular at $r=1$

$$\Psi_{\widehat{\Delta}}(r) = \frac{1}{2}\left(\widehat{g}_{\widehat{\Delta}}(r) + \frac{K_{p-\widehat{\Delta}}}{K_{\widehat{\Delta}}}\widehat{g}_{p-\widehat{\Delta}}(r)\right) = \frac{K_{p-\widehat{\Delta}}}{2K_p} r^{p-\widehat{\Delta}}\,{}_2F_1\left(\frac{p}{2},p-\widehat{\Delta},p,1-r^2\right)\,, \tag{3.42}$$

where we defined

$$K_{\widehat{\Delta}} = \frac{\Gamma(\widehat{\Delta})}{\Gamma\left(\widehat{\Delta}-\frac{p}{2}\right)}\,. \tag{3.43}$$

Also, the behavior of $\Psi_{\widehat{\Delta}}(r)$ near $r=1$ is such that the corresponding boundary term in (3.39) vanishes. However, near $r=0$ the functions grow as

$$\Psi_{\widehat{\Delta}}(r) \underset{r\to 0}{\sim} \frac{\Gamma(p-\widehat{\Delta})\Gamma\left(\widehat{\Delta}-\frac{p}{2}\right) r^{p-\widehat{\Delta}}}{2\Gamma(\widehat{\Delta})\Gamma\left(\frac{p}{2}-\widehat{\Delta}\right)} + \frac{r^{\widehat{\Delta}}}{2}\,, \tag{3.44}$$

and so at best they can be delta-function normalizable, provided we take $\mathrm{Re}(\widehat{\Delta}) = \frac{p}{2}$. Thus the eigenvalue spectrum of the operator $\mathcal{D}_{\|}$ is $\widehat{\Delta} = \frac{p}{2}+\mathrm{i}\nu\,,\nu\in\mathbb{R}$. Note that the delta function normalizability of the eigenfunctions is expected as the Sturm-Liouville problem in question is singular.

We shall now derive the (delta-function) orthonormality of these eigenfunctions explicitly. Let us first consider the following regularized functions that have the required boundary behavior as specified in eq. (3.40),

$$\Psi^{\text{reg.}}_{\widehat{\Delta}}(r) = \frac{K_{p-\widehat{\Delta}}}{2K_p} r^{p-\widehat{\Delta}+\epsilon}\,{}_2F_1\left(\frac{p}{2},p-\widehat{\Delta},p,1-r^2\right)\,, \qquad \text{with } \widehat{\Delta} = \frac{p}{2}+\mathrm{i}\,\nu\,, \quad \nu\in\mathbb{R}\,, \tag{3.45}$$

with $\epsilon>0$ a small number, such that the functions $\Psi^{\text{reg.}}_{\widehat{\Delta}}(r)$ are normalizable. The operator $\mathcal{D}_{\|}$ is self-adjoint on these functions, since the chosen regularization makes the boundary term at $r = 0$ vanish, while it preserves the vanishing of the boundary term at $r = 1$. However, due to the regularization $\Psi^{\text{reg.}}_{\widehat{\Delta}}(r)$ are not eigenfunctions of $\mathcal{D}_\|$. Nevertheless, starting from self-adjointness (the first line in (3.46)), we can evaluate the action of $\mathcal{D}_\|$ on the regularized functions to obtain

$$\begin{aligned} 0 &= \int_0^1 dr\, \mu_p(r) \left(\mathcal{D}_\|(\Psi^{\text{reg.}}_{\widehat{\Delta}_1}(r))\Psi^{\text{reg.}}_{\widehat{\Delta}_2}(r) - \Psi^{\text{reg.}}_{\widehat{\Delta}_1}(r)\mathcal{D}_\|(\Psi^{\text{reg.}}_{\widehat{\Delta}_2}(r))\right) \\ &= \left(\widehat{\Delta}_1(\widehat{\Delta}_1 - p) - \widehat{\Delta}_2(\widehat{\Delta}_2 - p)\right) \int_0^1 dr\, \mu_p(r)\, \Psi^{\text{reg.}}_{\widehat{\Delta}_1}(r)\Psi^{\text{reg.}}_{\widehat{\Delta}_2}(r) + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{3.46}$$

Taking $\epsilon \to 0$ this implies that if $\widehat{\Delta}_1(\widehat{\Delta}_1 - p) \neq \widehat{\Delta}_2(\widehat{\Delta}_2 - p)$ the functions are orthogonal. Finally, all we have to show now is what happens when the eigenvalues coincide, and for that we need only examine the behavior of the functions near $r = 0$ where the integral develops a singularity. In this case, taking $\widehat{\Delta}_i = p/2 + \mathrm{i}\,\nu_i$, we end up with integrals of the type

$$\int_0 dr\, r^{-1\pm\mathrm{i}(\nu_1\pm\nu_2)} = \pi\delta(\nu_1 \pm \nu_2) + \text{non-singular}\,, \tag{3.47}$$

following from the behavior of the measure (3.38) and (3.44). One could equivalently have shown that the integral of the regularized functions (3.45) provides a representation of the delta function as $\epsilon \to 0$. This is obvious for $p = 2$ wherein the resulting expressions are very simple.

All in all, the functions (3.42) are orthogonal when $\widehat{\Delta}_i = p/2 + \mathrm{i}\,\nu_i$, satisfying

$$\int_0^1 dr\, \mu_p(r)\, \Psi_{\widehat{\Delta}_1}(r)\Psi_{\widehat{\Delta}_2}(r) = \frac{\pi}{2}\frac{K_{p-\widehat{\Delta}_2}}{K_{\widehat{\Delta}_1}}\left[\delta(\nu_1 - \nu_2) + \delta(\nu_1 + \nu_2)\right]. \tag{3.48}$$

The functions $\Psi_{\widehat{\Delta}}(r)$ could be made real for $\nu \in \mathbb{R}$ by an appropriate choice of normalization, but we have not done so.

Angular factor of defect blocks

We now turn to the angular factor in the conformal block (3.36). It is useful to go back to the representation of the angular factor in (3.8) as a Gegenbauer polynomial for integer s via

$$w^{-s}{}_2F_1\left(-s, \frac{q}{2} - 1, 2 - \frac{q}{2} - s, w^2\right) = \binom{s + \frac{q}{2} - 2}{\frac{q}{2} - 2}^{-1} C_s^{q/2-1}\left(\frac{w}{2} + \frac{1}{2w}\right), \tag{3.49}$$

such that it becomes

$$\widehat{f}_s(\eta) = \binom{s + \frac{q}{2} - 2}{\frac{q}{2} - 2}^{-1} C_s^{(q/2-1)}(\eta)\,. \tag{3.50}$$

Gegenbauer polynomials satisfy the following orthogonality property,

$$\int_{-1}^1 d\eta\, \mu_q(\eta) C_s^{(\frac{q}{2}-1)}(\eta) C_{s'}^{(\frac{q}{2}-1)}(\eta) = \frac{2^{3-q}\pi\Gamma(s+q-2)}{\left(s + \frac{q}{2} - 1\right)\Gamma(s+1)\Gamma\left(\frac{q}{2} - 1\right)^2}\delta_{ss'}\,, \quad \mu_q(\eta) = (1-\eta^2)^{\frac{q-3}{2}}\,, \tag{3.51}$$

which we rewrite using the normalization of the conformal block themselves in the following manner,

$$\int_{-1}^{1} d\eta\, \mu_q(\eta)\widehat{f}_s(\eta)\widehat{f}_{s'}(\eta) = N_{q,s}\delta_{ss'}\,, \qquad N_{q,s} = 2^{3-q}\pi\frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma(s+\frac{q}{2})\Gamma\left(s+\frac{q}{2}-1\right)}\,. \tag{3.52}$$

Note that the orthonormality of the Gegenbauer polynomials is on the real interval $[-1,1]$ (and with respect to the measure specified) and this range of values of η is what makes the inversion formula that we are seeking essentially Euclidean.

The inversion formula

From the previous discussion on orthogonal functions which provide a complete basis for two different Hilbert spaces of functions (corresponding to the radial and angular parts of the defect channel conformal blocks), we can write the following Euclidean partial wave expansion for the two-point function. This is simply a completeness relation, analogous to the Euclidean partial wave expansion for a four-point function in CFTs without defects [166] (see also [82]). To establish this more rigorously, we should treat the problem from the point of view of harmonic analysis on the $SO(p+1,1)\times SO(q)$ symmetry group of the Euclidean theory (see [167] and [206]).

$$\mathcal{A}(r,\eta) = \sum_{s=0}^{\infty}\int_{\gamma}\frac{d\widehat{\Delta}}{2\pi \mathrm{i}}b(\widehat{\Delta},s)\widehat{f}_s(\eta)\Psi_{\widehat{\Delta}}(r), \qquad \gamma = \{\widehat{\Delta} : \widehat{\Delta}\in(p/2-\mathrm{i}\infty, p/2+\mathrm{i}\infty)\}\,. \tag{3.53}$$

Since $\Psi_{p-\widehat{\Delta}}(r) = \frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}\Psi_{\widehat{\Delta}}(r)$, we have,

$$b(p-\widehat{\Delta},s) = \frac{K_{p-\widehat{\Delta}}}{K_{\widehat{\Delta}}}b(\widehat{\Delta},s)\,. \tag{3.54}$$

The position of the poles and residues of $b(\widehat{\Delta},s)$ is revealed by closing the contour γ. At large (real) $\widehat{\Delta}$

$$\widehat{g}_{\widehat{\Delta}}(r) \sim r^{\widehat{\Delta}}(1-r^2)^{-p/2}\,, \tag{3.55}$$

so the contour must be closed to the right on the first addend in $\Psi_{\widehat{\Delta}}$, and to the left on the second - see eq. (3.42). In order for the result to agree with the usual conformal block decomposition, $b(\widehat{\Delta},s)$ must have single poles in correspondence with the spectrum, and the residue must coincide, up to a sign, with the OPE coefficient:

$$\mathcal{A}(r,\eta) = \sum_{s=0}^{\infty}\widehat{f}_s(\eta)\sum_{\widehat{\Delta}^*\in\text{spectrum}} b^2_{s,\widehat{\Delta}^*}\widehat{g}_{\widehat{\Delta}^*}(r)\,, \qquad b^2_{s,\widehat{\Delta}^*} = -\mathrm{Res}_{\widehat{\Delta}=\widehat{\Delta}^*}b(\widehat{\Delta},s)\,. \tag{3.56}$$

For defect operators of dimension less than $p/2$ we must deform the contour such that it picks up the pole on the left and does not pick up the reflection according to (3.54) on the right. Similarly if the operator has dimension exactly $p/2$ we must take the principle-value of the integral to pick up half of the residue.

Not all poles in (3.53) arise from poles of $b(\widehat{\Delta}, s)$, as the defect blocks themselves have poles for special values of $\widehat{\Delta}$ and s. However, since the defect blocks $\widehat{g}_{\widehat{\Delta}}(r)$ have poles for $\widehat{\Delta} = \frac{p}{2} - n$ [65] they are always to the left of $\frac{p}{2}$, and thus are not picked up when we close the contour to the right. Similarly, for the second addend in $\Psi_{\widehat{\Delta}}$, we close the contour to the left while $\widehat{g}_{p-\widehat{\Delta}}(r)$ only has poles to the right of $\frac{p}{2}$.

Eq. (3.53) can be easily inverted using the orthogonality relations (3.48) and (3.52), yielding the following Euclidean inversion formula[3]

$$b(\widehat{\Delta}, s) = \frac{2}{N_{q,s}} \frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}} \int_{-1}^{1} d\eta \int_{0}^{1} dr\, \mu_p(r) \mu_q(\eta)\, \widehat{f}_s(\eta) \Psi_{\widehat{\Delta}}(r) \mathcal{A}(r, \eta)\,. \tag{3.57}$$

η takes values on the interval $[-1, 1]$ and is just the cosine of an angle (see (3.35)). Thus we change variables in the above integral to obtain

$$\begin{aligned} b(\widehat{\Delta}, s) &= \frac{1}{N_{q,s}} \frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}} \oint\limits_{|w|=1} \frac{dw}{\mathrm{i}w} \int_{0}^{1} dr\, \mu(r, w)\, \widehat{f}_s \left(\frac{1}{2w} + \frac{w}{2} \right) \Psi_{\widehat{\Delta}}(r)\, \mathcal{A}\,(r, w)\,, \\ \mu(r, w) &= \mu_p(r) \left| \frac{w}{2\mathrm{i}} - \frac{1}{2\mathrm{i}w} \right|^{q-2} . \end{aligned} \tag{3.58}$$

$\mathcal{A}(r, w)$ is the correlator as a function of r and w. w is a phase and this reflects the fact that the inversion formula in eq. (3.57) involves Euclidean kinematics. The angular part of the measure is $\left[\left(\frac{w}{2\mathrm{i}} - \frac{1}{2\mathrm{i}w} \right)^2 \right]^{\frac{q-2}{2}}$ which is equal to $\left| \frac{w}{2\mathrm{i}} - \frac{1}{2\mathrm{i}w} \right|^{q-2}$ when w is a phase. Later when we deform the w contour, we shall use the former expression for the angular part of the measure $\mu(r, w)$. Note that the measure thus has a branch cut for odd q but not for even q.

The above integral however might not converge depending on the behavior of the correlator at the end points. Divergences at $r = 0$ may arise depending on $\widehat{\Delta}$. For instance, let us choose $\widehat{\Delta} > p/2$. Then the kernel of eq. (3.58) has an expansion in growing powers of r that starts with $r^{-1-\widehat{\Delta}}$. On the other hand, unitarity guarantees that $\mathcal{A}(r, \eta)$ is regular. One can then separately integrate the divergent terms in the small r expansion of the integrand in the inversion formula (3.58), defining the result by analytic continuation in $\widehat{\Delta}$,

$$\int_0^1 dr\, r^{-1-\widehat{\Delta}+\alpha} = \frac{1}{\alpha - \widehat{\Delta}}\,, \tag{3.59}$$

and then add it back into the right hand side of the inversion formula. In particular, if a primary of dimension $\widehat{\Delta}^*$ is present, $\mathcal{A}(r, \eta)$ contains a term $r^{\widehat{\Delta}^*}$, which precisely provides the expected pole in the OPE coefficient function $b(\widehat{\Delta}, s)$.

Divergences at $r = 1$, on the other hand, are controlled by the bulk channel OPE, and may be regulated by cutting off the integral at $r \leq 1 - \epsilon$. The kernel $\mu(r, w)\Psi_{\widehat{\Delta}}(r)$ has a regular Taylor expansion close to $r = 1$, therefore the divergent part of the inversion formula as $\epsilon \to 0$ does not contain poles in $\widehat{\Delta}$. Hence, it can be safely dropped without altering the spectrum and the residues of $b(\widehat{\Delta}, s)$.

[3] This formula was derived in collaboration with D. Mazáč.

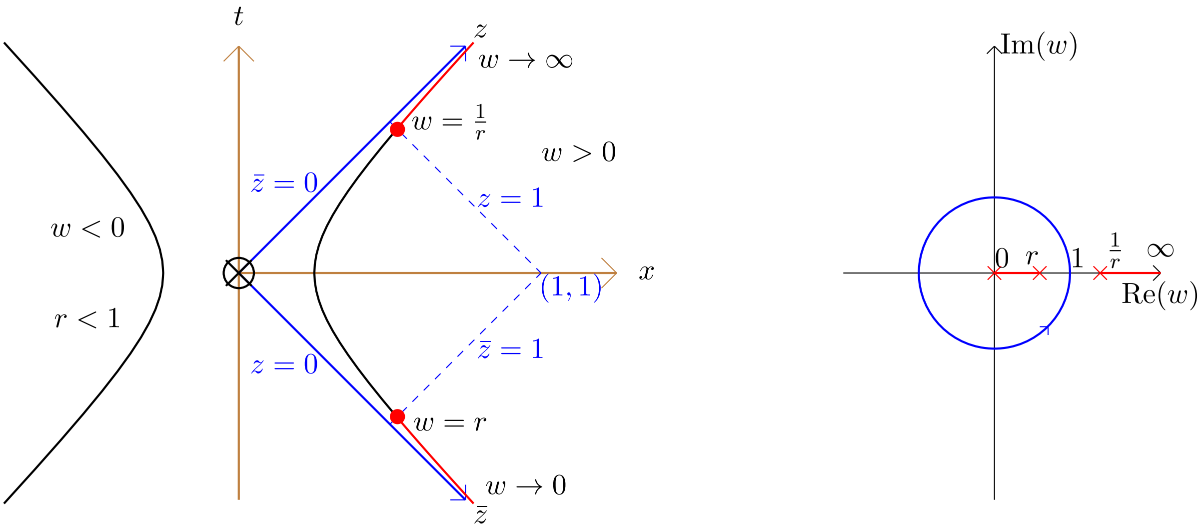


Figure 3.3: The positive real axis on the complex w plane, at fixed $r < 1$, maps to the black solid hyperbola in the $(z, \bar{z})$ plane in the figure on the left. The bulk OPE singularities denoted by red dots, correspond to the intersection of the hyperbola with the past and future lightcones of the operator $\phi(1,1)$. The parts of the hyperbola marked in red correspond to configurations where the operator $\phi(z, \bar{z})$ is causally separated from the operator $\phi(1,1)$. On the complex w plane there are branch cuts extending over these values of $\mathrm{Re}(w)$ as shown (in red) in the figure on the right. On the left half plane in the figure on the left, the entire hyperbola is spacelike separated from the operator $\phi(1,1)$. Hence there are no branch cuts for negative $\mathrm{Re}(w)$ in the figure on the right.

To conclude the section, let us just state that eq. (3.58) is the Euclidean inversion formula that we wanted to derive. Note that this formula makes sense only for integer values of s as the angular part of the block $\widehat{f}_s$ is defined as in eq. (3.50) only for integer s.

3.4.2 The Lorentzian formula

In the Euclidean inversion formula (3.58) the contour of integration in the complex w plane is the unit circle, as w is a phase. We now want to deform the contour in order to integrate over real values of w, which corresponds to a Lorentzian configuration. We need to consider the w dependence of the correlator and also the w dependence in the angular blocks $\widehat{f}_s$ and the measure μ_p. First let us have a discussion on the singularities of the Lorentzian correlator on the complex w plane.

The Lorentzian correlator on the complex w plane

The range of r in the Euclidean formula is confined between 0 and 1. All other points in the Euclidean plane are related to this fundamental region by inversion. In fact the $(z, \bar{z})$ is twice mapped onto the (r, w) plane. $r < 1$ and $r > 1$ correspond to each of these copies of the $(z, \bar{z})$ plane. In Euclidean kinematics, the distance (squared) between the two operators (or to the defect) is positive definite and the correlator is always analytic away from the coincident points corresponding to the OPE singularities.

As it is clear from fig. 3.3, at fixed $r < 1$ the correlator $\mathcal{A}(r, w)$ has two copies of the bulk OPE singularity at $w = r$ and $w = 1/r$ when the operators are lightlike separated. There is no singularity at negative values of w as the operators are then spacelike separated. Further singularities may lie in the limits $w = 0$ and $w = \infty$, which are double lightcone limits. While the correlator is single valued on the circle $|w| = 1$ in the complex w plane, the OPE singularities at $w = r$ and $w = \frac{1}{r}$ are branch points. The cuts run from 0 to r and from $1/r$ to ∞. Let us now understand the physical meaning of these branch cuts.

Let us first consider a two-point function in a CFT without defects in Euclidean kinematics, restricting to two dimensions for notational simplicity.

$$\mathcal{G}(\tau, x) = \frac{1}{(\tau^2 + x^2)^{\frac{\Delta}{2}}} . \tag{3.60}$$

To continue this correlator to Lorentzian kinematics, we have to take $\tau = \epsilon + \mathrm{i}t$ and then take the limit $\epsilon \to 0$. This give us, to $O(\epsilon^2)$,

$$\mathcal{G}(\epsilon + \mathrm{i}t, x) = \frac{1}{(2\mathrm{i}\epsilon t - t^2 + x^2)^{\frac{\Delta}{2}}} . \tag{3.61}$$

From eq. (3.61), we can already see that the Lorentzian correlator can be expected to have branch cuts for generic values of the dimension Δ as it is no longer positive definite. Let us now consider $\mathcal{G}(y)$ on the complex y plane. When $x^2 - t^2 > 0$, the operators are spacelike separated and the limit $\lim_{\epsilon \to 0} \mathcal{G}(\epsilon + \mathrm{i}t, x)$ is unique. However when $x^2 - t^2 < 0$, the limit $\lim_{\epsilon \to 0} \mathcal{G}(\epsilon + \mathrm{i}t, x)$ does not exist as the limiting value(s) depends on the sign of ϵ. This is explained in fig. 3.4. One of these limiting values corresponds to the time ordered Lorentzian correlator while the other one corresponds to the anti-time ordered Lorentzian correlator.

The discontinuity accross the branch cut in the Lorentzian correlator is just the reflection of the fact that causally separated operators in a Lorentzian theory do not commute. Translating this discussion to momentum space would just give us the familiar discussion on Feynman's iϵ prescription where the sign of ϵ determines the contour and thus the direction of the Wick rotation from the Euclidean to the Lorentzian correlation function (such that we do not cross any poles) which gives us either the time ordered or the anti-time ordered two-point function.

This story easily generlizes to correlation functions of more than two operators where the analytic continuation of the Euclidean correlator to Lorentzian kinematics gives us a correlator with an intricate structure of singularities and branch cuts with the different branches corresponding to the time ordered, anti-time ordered and mixed order correlators (see sec. 3.1 of [193] for a nice discussion on the topic.)

Now coming back to the bulk two-point function in the defect CFT, we have a straightforward generalization of the previous discussion. The Euclidean correlator is single-valued and has bulk OPE singularities at $w = r$ and $w = \frac{1}{r}$. Continuing the correlator beyond these singularities to Lorentzian kinematics can possibly land it on different branches as they do

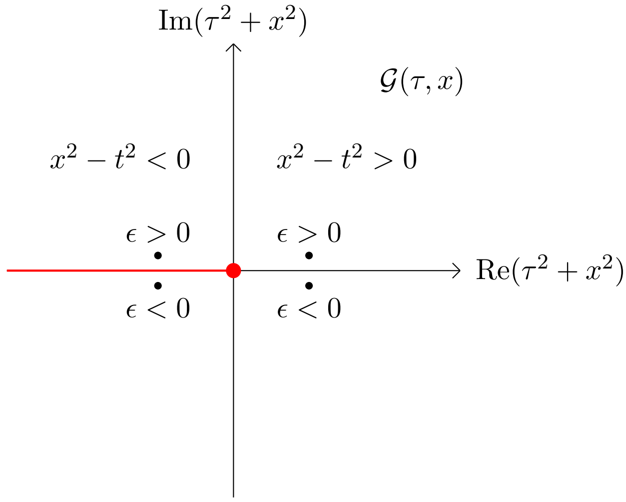


Figure 3.4: The red dot at the origin is the coincident point OPE singularity and the red line on the negative real axis is the branch cut as the continuation of the Euclidean correlator to Lorentzian kinematics takes us onto two different branches, depending on the sign of ϵ, when the operators are timelike separated.

not commute when causally separated. This gives us the two branch cuts along real w from $(0, r)$ and $(\frac{1}{r}, \infty)$.

w dependence of Angular blocks

Now we want to consider the angular blocks $\widehat{f}_s$ and the corresponding w dependence in the Euclidean inversion formula (3.58). These are solutions to the eigenvalue equation of the quadratic Casimir of the group of rotations around the defect. It is useful to consider a larger set of solutions to this Casimir equation as we will be able to use the asymptotics of these solutions effectively in the contour deformation procedure that will give us the Lorentzian formula.

We make use of (3.49) again to go back to a representation of $\widehat{f}_s(\eta)$ (3.50) as a hypergeometric function

$$\widehat{h}_1(s, w) = \binom{s + \frac{q}{2} - 2}{\frac{q}{2} - 2}^{-1} C_s^{q/2-1}\left(\frac{w}{2} + \frac{1}{2w}\right) = w^{-s}{}_2F_1\left(-s, \frac{q}{2} - 1, 2 - \frac{q}{2} - s, w^2\right) . \quad (3.62)$$

Recall that when q is even, an order of limits ambiguity arises in the definition of the hypergeometric function. The equality (3.49) holds if we first take s to be integer, and then q to be even. As before, this prescription assumed every time it is necessary.

Other solutions to the Casimir equation for the angular part of the defect blocks can be obtained by combining the transformations $w \to 1/w$ - which leaves $\widehat{h}_1$ invariant when s is integer - and $s \to 2 - q - s$, both of which are symmetries of the Casimir equation. We will use the two following solutions

$$\widehat{h}_2(s,w) := \widehat{h}_1(2-q-s,w) = w^{s+q-2}{}_2F_1\left(s+q-2,\frac{q}{2}-1,\frac{q}{2}+s,w^2\right), \tag{3.63}$$

$$\widehat{h}_3(s,w) := \widehat{h}_2(s,1/w) = w^{2-q-s}{}_2F_1\left(s+q-2,\frac{q}{2}-1,\frac{q}{2}+s,\frac{1}{w^2}\right), \tag{3.64}$$

where $\widehat{h}_2$ is regular in the origin while $\widehat{h}_3$ is regular at infinity. Ideally, one would like to express $\widehat{h}_1$ as a linear combination of $\widehat{h}_2$ and $\widehat{h}_3$, but this is globally possible only for defects of even codimension. When q is even, the discontinuities of $\widehat{h}_2$ and $\widehat{h}_3$ vanish. Let us first consider this simpler case.

Even q

As mentioned before our strategy is to take the Euclidean inversion formula in eq. (3.58) and deform the contour of the w integral to Lorentzian kinematics. In particular, we shall obtain a formula that will express $b(\widehat{\Delta},s)$ as an integral over the discontinuity in the correlator across the branch cuts running from $w=0$ to $w=r$ and from $w=\frac{1}{r}$ to $w\to\infty$.

$$\operatorname{Disc}\mathcal{A}(r,w) = \mathcal{A}(r,w+\mathrm{i}0) - \mathcal{A}(r,w-\mathrm{i}0)\,. \tag{3.65}$$

To deform the w contour such that it wraps around the positive real axis, we will have to drop arcs around $w=\infty$ and $w=0$. In order to do that, we have to ensure the regularity of the integrand in eq. (3.58) in these extremities. Let us first look at the angular blocks $\widehat{f}_s$. In the case of even codimension, we have (see appendix. B for the derivation),

$$\widehat{f}_s \equiv \widehat{h}_1(s,w) = (-1)^{\frac{q}{2}-1}\frac{\Gamma(s+1)\Gamma\left(s+q-2\right)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(s+\frac{q}{2}\right)}\left(\widehat{h}_2(s,w)+\widehat{h}_3(s,w)\right), \qquad s=0,1,2\ldots, \quad q=2,4,6,\ldots. \tag{3.66}$$

Let us recall that $\widehat{h}_2$ damps off at the origin and $\widehat{h}_3$ damps off at infinity. Therefore, after plugging eq. (3.66) in the Euclidean inversion formula (3.58), we can deform the contour towards the interior on $\widehat{h}_2$ such that it wraps around the discontinuity from $w=0$ to $w=r$, and towards the exterior on $\widehat{h}_3$ such that it wraps around the discontinuity from $w=\frac{1}{r}$ to $w\to\infty$ (see fig. 3.5). As mentioned before, we are now using $\left[\left(\frac{w}{2\mathrm{i}}+\frac{1}{2\mathrm{i}w}\right)^2\right]^{\frac{q-2}{2}}$ as the angular part of the measure and for even q this does not have any branch cuts.

We still need to ensure that the growth of the correlator itself is compatible with the contour defomations. When deforming the contour towards the interior, divergences may arise when shrinking the circles around $w=0$ and $w=r$. The former is not an OPE limit. $w=0$ lies at the boundary of the region of convergence of the bulk channel OPE, and the bulk channel expansion lacks positivity. In the case of the Caron-Huot formula [74] discussed in sec. 3.1, positivity of the block expansion was used to place a bound on the growth of the correlator in the Regge limit, which has the same role there as the small w limit here. Deprived of this tool, we currently have no way to constrain the growth of the correlator in general.

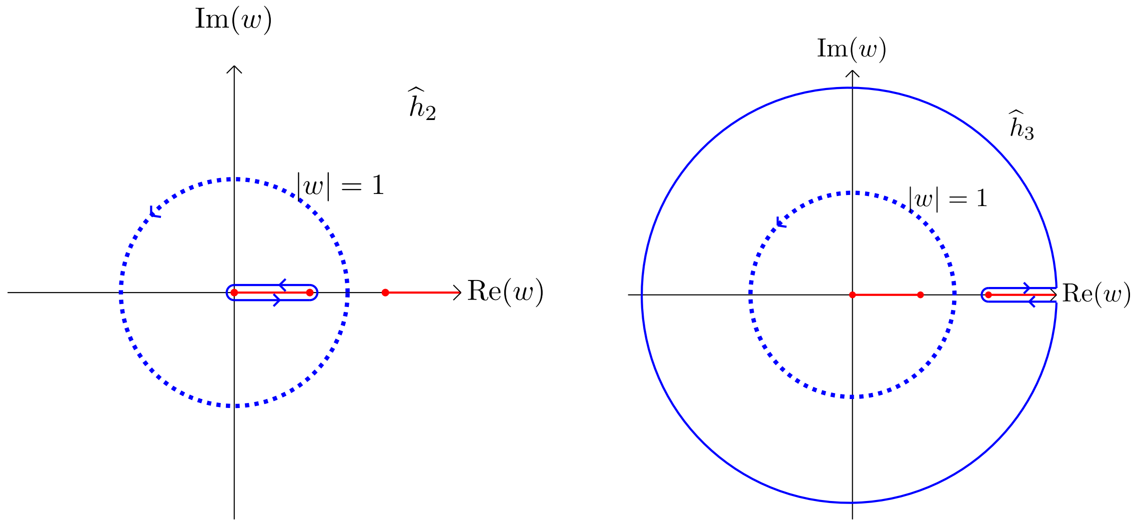


Figure 3.5: Lorentzian inversion formula for even q: The term with $\widehat{h}_2$ as the angular block has a contour deformation on the Euclidean formula in eq. (3.58) as shown on the left and the term with $\widehat{h}_3$ as the angular block has a contour deformation as shown on the right. The branch cuts are marked in red, the old contour corresponding to Euclidean kinematics dotted in blue and the new contours in blue.

If for small w the two-point function is bounded by a power, then for s large enough the circle around $w = 0$ can be shrunk. More precisely

$$\text{if } \mathcal{A}(r, w) \lesssim w^{-s_\star}, \quad \text{as } w \to 0, \qquad \text{then the formula is valid for } s > s_\star, \tag{3.67}$$

since the integrand in (3.58) is then bounded by $w^{s-1-s_\star}$ for $w \to 0$. For the trivial defect correlation function (3.11), the formula converges to spin $s = 0$.

The point $w = r$ is a bulk OPE singularity, that includes terms with power law behavior of the kind $(r - w)^{-\Delta_\phi + \tau/2}$. The integral converges for negative enough Δ_ϕ, and can then be analytically continued. The procedure is allowed because the angular integral in the original Euclidean formula is convergent for all values of Δ_ϕ. This takes care of the contour deformation around the branch cut from $w = 0$ to $w = r$ (see fig. 3.5) for the term with $\widehat{h}_2$ as the angular factor.

We can proceed in the same way for $\widehat{h}_3$, now deforming the contour towards the exterior. The end points of the branch cuts are related by inversion of w, hence the same discussion holds true.

Applying the described contour deformation on eq. (3.58), we evetually obtain the following Lorentzian formula:

$$
\begin{aligned}
b(\widehat{\Delta}, s) &= \frac{1}{2}\frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}\left(b_0(\widehat{\Delta}, s) + b_\infty(\widehat{\Delta}, s)\right), \\
b_0(\widehat{\Delta}, s) &= -\int_0^1 dr \int_0^r \frac{dw}{\mathrm{i}\pi w} w^{2-q}(1-w^2)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_2(s,w)\Psi_{\widehat{\Delta}}(r)\,\mathrm{Disc}\,\mathcal{A}(r,w)\,, \\
b_\infty(\widehat{\Delta}, s) &= \int_0^1 dr \int_{1/r}^\infty \frac{dw}{\mathrm{i}\pi w} w^{2-q}(w^2-1)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_3(s,w)\Psi_{\widehat{\Delta}}(r)\,\mathrm{Disc}\,\mathcal{A}(r,w)\,.
\end{aligned}
\tag{3.68}
$$

Odd q

When the codimension is odd, the procedure to be adopted is slightly more involved. The relation (3.66), which was a key ingredient of the contour deformation procedure in the case of an even codimension, cannot be globally valid now, because of the cuts in $\widehat{h}_2$ and $\widehat{h}_3$. We shall therefore resort to the following trick: we shall add zero in the form of a contour integral over $\widehat{h}_3$ along the contours C_+ and C_- as shown in fig. 3.6.

$$
b(\widehat{\Delta}, s) = b(\widehat{\Delta}, s) + c_+ \oint_{C_+} dw\,[\ldots]\widehat{h}_3(s,w) + c_- \oint_{C_-} dw\,[\ldots]\widehat{h}_3(s,w)\,. \tag{3.69}
$$

where the dots stand for everything in eq. (3.58) except $\widehat{f}_s(\eta)$. The idea is to choose $c_\pm$ such that the linear combinations of $\widehat{h}_1$ in $b(\widehat{\Delta}, s)$ and $\widehat{h}_3$ of the added contour integrals result in an integrand that is regular near the origin and at infinity. For now, let us state the values of $c_\pm$ and we shall show later how these particular values give the desired effect.

$$
c_\pm = -e^{\pm \mathrm{i}\frac{\pi}{2}q}\frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(s+\frac{q}{2}\right)}\,. \tag{3.70}
$$

Notice that the combination is different in the upper and lower plane. It should be clear by the end of this discussion that this is necessitated by the presence of a cut in $\widehat{h}_3$, which extends from $w=-1$ to $w=1$.

The structure of cuts in the integrand is complicated by the contribution of $\mu(r,w)$ - see eq. (3.58). As an analytic function of w, the angular measure factor $\tilde{\mu}_q(w) = \left[\left(\frac{w}{2\mathrm{i}} - \frac{1}{2\mathrm{i}w}\right)^2\right]^{\frac{q-2}{2}}$ has a branch cut for odd q. One can show that, for $w = x + \mathrm{i}\epsilon$ with $x, \epsilon \in \mathbb{R}$,

$$
\left(\frac{w}{2\mathrm{i}} - \frac{1}{2\mathrm{i}w}\right)^2 = -\frac{1}{4}\left[\left(x-\frac{1}{x}\right)^2 + 2\mathrm{i}\epsilon x\left(1-\frac{1}{x^4}\right)\right] + O\left(\epsilon^2\right)\,. \tag{3.71}
$$

Therefore the branch cut in the angular measure factor $\tilde{\mu}_q(w)$ is the entire real axis. From eq. (3.71), one obtains that (with $w \in \mathbb{R}$),

$$
\tilde{\mu}_q(w \pm \mathrm{i}0) = \begin{cases}
-\left[\frac{1}{2}\left|w-\frac{1}{w}\right|\right]^{q-2} e^{\pm \mathrm{i}q\frac{\pi}{2}} & w < -1\,, \\
-\left[\frac{1}{2}\left|w-\frac{1}{w}\right|\right]^{q-2} e^{\mp \mathrm{i}q\frac{\pi}{2}} & -1 < w < 0\,, \\
-\left[\frac{1}{2}\left|w-\frac{1}{w}\right|\right]^{q-2} e^{\pm \mathrm{i}q\frac{\pi}{2}} & 0 < w < 1\,, \\
-\left[\frac{1}{2}\left|w-\frac{1}{w}\right|\right]^{q-2} e^{\mp \mathrm{i}q\frac{\pi}{2}} & w > 1\,.
\end{cases} \tag{3.72}
$$

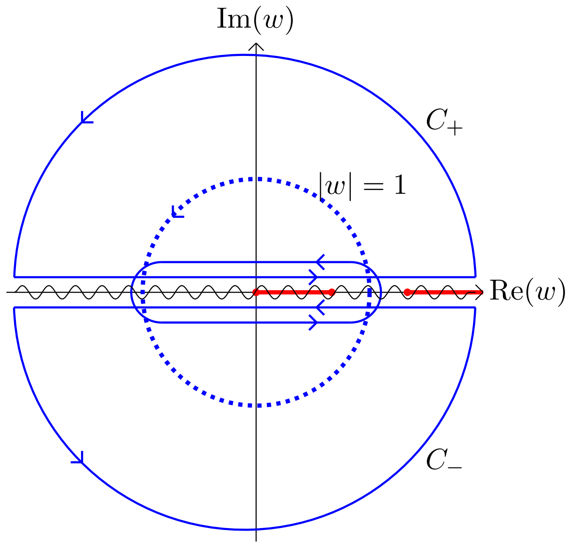


Figure 3.6: Lorentzian formula for odd q: Zeroes are added in the form of contour integrals over $C_\pm$ shown in blue. The $|w| = 1$ contour from the Euclidean formula, dotted in blue, is deformed to wrap around the real axis from $w = -1$ to $w = 1$ and overlap with $C_\pm$. The curvy line along the real axis denote the branch cut in the measure $\tilde{\mu}_q(w)$ while the red lines denote the discontinuity in the correlator $\mathcal{A}$.

Now we are ready for the contour deformation procedure. As shown in fig. 3.6, we shall deform the contour of integration in $b(\widehat{\Delta}, s)$ towards the real axis such that it wraps around the real line from $w = -1$ to $w = 1$ and there is overlap with the contour C_+ on the upper half plane and with C_- on the lower half plane.

Since $\widehat{h}_3$ damps off at infinity, we can drop the arcs of $C_\pm$ at infinity. What remains to do is to analyze what happens along the real w axis. Let us consider the following four intervals separately: $w < -1$, $-1 < w < 0$, $0 < w < 1$ and $w > 1$ with $w \in \mathbb{R}$.

When $w < -1$, we have the following integral from eq. (3.69),

$$\frac{1}{N_{q,s}} \frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}} \int_{-\infty}^{-1} \frac{dw}{iw} \int_0^1 dr \, \mu_p(r) \Psi_{\widehat{\Delta}}(r) \left[c_+ \tilde{\mu}_q(w + i0) - c_- \tilde{\mu}_q(w - i0) \right] \widehat{h}_3(s, w) \mathcal{A}(r, w) \,. \tag{3.73}$$

$\widehat{h}_3(s, w)$ and the correlator $\mathcal{A}(r, w)$ have no cuts in this region (hence we have pulled them out of the brackets in eq. (3.73)). Notice from eq. (3.70) and eq. (3.72) that the discontinuity in $\tilde{\mu}_q(w)$ across the real line offsets the difference between c_+ and c_-. Thus the integral in eq. (3.73) evaluates to 0 and we have no contribution from the region $w < 1$.

Now we come to the interval $-1 < w < 0$. Let us first note the following identities (see appendix. B for the derivation) which are analogous to the one in eq. (3.66):

$$\begin{aligned}\widehat{h}_1(s,w)-c_+\widehat{h}_3(s,w)&=-e^{-\mathrm{i}\frac{\pi}{2}q}\frac{\Gamma(s+1)\Gamma\left(s+q-2\right)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(s+\frac{q}{2}\right)}\widehat{h}_2(s,w)\,, \quad \mathrm{Im}(w)>0\,,\ |\mathrm{Re}(w)|<1\,,\\ \widehat{h}_1(s,w)-c_-\widehat{h}_3(s,w)&=-e^{\mathrm{i}\frac{\pi}{2}q}\frac{\Gamma(s+1)\Gamma\left(s+q-2\right)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(s+\frac{q}{2}\right)}\widehat{h}_2(s,w)\,, \quad \mathrm{Im}(w)<0\,,\ |\mathrm{Re}(w)|<1\,.\end{aligned} \tag{3.74}$$

From eq. (3.69), we get the following integral for this region,

$$\begin{aligned}-\frac{1}{N_{q,s}}\frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}\int_{-1}^{0}\frac{dw}{\mathrm{i}w}\int_0^1 dr\,\mu_p(r)\Psi_{\widehat{\Delta}}(r)\,\Big[\tilde{\mu}_q(w+\mathrm{i}0)\left(\widehat{h}_1(w+\mathrm{i}0)-c_+\widehat{h}_3(s,w+\mathrm{i}0)\right)\\ -\tilde{\mu}_q(w-\mathrm{i}0)\left(\widehat{h}_1(w-\mathrm{i}0)-c_-\widehat{h}_3(s,w-\mathrm{i}0)\right)\Big]\,\mathcal{A}(r,w)\,.\end{aligned} \tag{3.75}$$

The correlator $\mathcal{A}(r,w)$ does not have any branch cuts in this region. Using the identities in eq. (3.74) and the expression for $N_{q,s}$ in eq. (3.52), we get,

$$\frac{2^{q-3}}{\pi}\frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}\int_{-1}^{0}\frac{dw}{\mathrm{i}w}\int_0^1 dr\,\mu_p(r)\Psi_{\widehat{\Delta}}(r)\,\left[\tilde{\mu}_q(w+\mathrm{i}0)e^{-\mathrm{i}\frac{\pi}{2}q}-\tilde{\mu}_q(w-\mathrm{i}0)e^{\mathrm{i}\frac{\pi}{2}q}\right]\,\widehat{h}_2(s,w)\mathcal{A}(r,w)\,. \tag{3.76}$$

This is again equal to 0 and thus we get no contribution from the negative real values of w.

Now we come to the region $0<w<1$. The integral from $w=r$ to $w=1$ is identical to the previous case of $-1<w<0$ and we receive zero contribution. The integral from $w=0$ to $w=r$ is different as the correlator $\mathcal{A}(r,w)$ has a discontinuity across the real line from $w=0$ to $w=r$. From eq. (3.69), (and using eq. (3.74)) we get,

$$\begin{aligned}\frac{1}{N_{q,s}}\frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}\int_0^r\frac{dw}{\mathrm{i}w}\int_0^1 dr\,\mu_p(r)\Psi_{\widehat{\Delta}}(r)\,\Big[\tilde{\mu}_q(w+\mathrm{i}0)e^{-\mathrm{i}\frac{\pi}{2}q}\mathcal{A}(w+\mathrm{i}0)\\ -\tilde{\mu}_q(w-\mathrm{i}0)e^{\mathrm{i}\frac{\pi}{2}q}\mathcal{A}(w-\mathrm{i}0)\Big]\,\frac{\Gamma(s+1)\Gamma\left(s+q-2\right)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(s+\frac{q}{2}\right)}\widehat{h}_2(s,w)\,.\end{aligned} \tag{3.77}$$

Once again, the difference between c_+ and c_- nicely cancels the difference between $\mu_q(w+\mathrm{i}0)$ and $\mu_q(w-\mathrm{i}0)$ in eq. (3.77) to give us,

$$-\frac{K_{\widehat{\Delta}}}{2K_{p-\widehat{\Delta}}}\int_0^1 dr\int_0^r\frac{dw}{\mathrm{i}\pi w}\,w^{2-q}(1-w^2)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_2(r,w)\Psi_{\widehat{\Delta}}(r)\mathrm{Disc}\,\mathcal{A}(r,w). \tag{3.78}$$

Finally let us consider the region $|w|>1$. The contribution from $w=1$ to $w=\frac{1}{r}$ to the contour integral in eq. (3.69) is zero as there is no discontinuity in the correlator or $\widehat{h}_3(s,w)$ in this region. The difference between c_+ and c_- is adjusted with the discontinuity in the measure $\tilde{\mu}_q(w)$ across the real axis. The correlator has a cut from from $w=\frac{1}{r}$ to $w\to\infty$. The contribution to eq. (3.69) from this region can be simply obtained by plugging in eq. (3.70) and eq. (3.72) into eq. (3.69). We get,

$$\frac{K_{\widehat{\Delta}}}{2K_{p-\widehat{\Delta}}}\int_0^1 dr\int_{\frac{1}{r}}^{\infty}\frac{dw}{\mathrm{i}\pi w}\,w^{2-q}(w^2-1)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_3(r,w)\Psi_{\widehat{\Delta}}(r)\mathrm{Disc}\,\mathcal{A}(r,w). \tag{3.79}$$

Combining all the contributions to the contour integral in eq. (3.69) all together, we obtain a Lorentzian inversion formula identical to the case of even codimension, namely eq. (3.68).

Lorentzian inversion formula for codimension q

Thus, we obtain the following Lorentzian formula, valid for both even and odd q,

$$\begin{aligned} b(\widehat{\Delta}, s) &= \frac{1}{2}\frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}(b_0(\widehat{\Delta}, s) + b_\infty(\widehat{\Delta}, s))\,, \\ b_0(\widehat{\Delta}, s) &= -\int_0^1 dr \int_0^r \frac{dw}{\mathrm{i}\pi w} w^{2-q}(1-w^2)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_2(s,w)\Psi_{\widehat{\Delta}}(r)\mathrm{Disc}\,\mathcal{A}(r,w)\,, \\ b_\infty(\widehat{\Delta}, s) &= \int_0^1 dr \int_{1/r}^\infty \frac{dw}{\mathrm{i}\pi w} w^{2-q}(w^2-1)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_3(s,w)\Psi_{\widehat{\Delta}}(r)\mathrm{Disc}\,\mathcal{A}(r,w)\,. \end{aligned} \tag{3.80}$$

Note that in any theory with $q > 2$ the two-point function obeys $\mathcal{A}(r, w) = \mathcal{A}(r, \frac{1}{w})$, and thus it follows that $\mathrm{Disc}\,\mathcal{A}(r, w) = -\mathrm{Disc}\,\mathcal{A}(r, 1/w)$. In the case $q = 2$ the symmetry of the two-point function is only present in a parity invariant theory, which is assumed here since we have taken the blocks in eq. (3.50) to be symmetric under $w \to \frac{1}{w}$. Therefore, applying this inversion in w to eq. (3.68), we find that,

$$b_0(\widehat{\Delta}, s) = b_\infty(\widehat{\Delta}, s)\ . \tag{3.81}$$

This simplifies eq. (3.68) and we obtain,

$$b(\widehat{\Delta}, s) = -\frac{K_{\widehat{\Delta}}}{K_{p-\widehat{\Delta}}}\int_0^1 dr \int_0^r \frac{dw}{\mathrm{i}\pi w} w^{2-q}(1-w^2)^{q-2}(1-r^2)^p r^{-p-1}\widehat{h}_2(s,w)\Psi_{\widehat{\Delta}}(r)\mathrm{Disc}\mathcal{A}(r,w)\,. \tag{3.82}$$

Recall that $\Psi_{\widehat{\Delta}}(r)$ has one term corresponding to the physical operator and one corresponding to it's shadow:

$$\Psi_{\widehat{\Delta}}(r) = \frac{1}{2}\left(\widehat{g}_{\widehat{\Delta}}(r) + \frac{K_{p-\widehat{\Delta}}}{K_{\widehat{\Delta}}}\widehat{g}_{p-\widehat{\Delta}}(r)\right) \tag{3.83}$$

The first term in the sum $\widehat{g}_{\widehat{\Delta}}(r)$ goes as $r^{\widehat{\Delta}}$ as $r \to 0$ and the poles it contributes to $b(\widehat{\Delta}, s)$ correspond to the shadow operators rather than the physical operators in the defect spectrum. Hence we need only keep the second addend in $\Psi_{\widehat{\Delta}}(r)$. In the next section, we shall show how to apply the Lorentzian inversion formula to obtain the spectrum. Following these examples, one can use the inversion formula keeping only the first addend of $\Psi_{\widehat{\Delta}}(r)$ to explicitly check that the first addend contributes to poles corresponding to the shadow operators in the spectrum. We shall however continue using the notation $b(\widehat{\Delta}, s)$ although it has only the poles corresponding to the physical spectrum. Thus we obtain,

$$\begin{aligned} b(\widehat{\Delta},s) &= -\int_0^1 dr \int_0^r \frac{dw}{\mathrm{i}\pi w} w^s (1-w^2)^{q-2}(1-r^2)^p r^{-\widehat{\Delta}-1}\, {}_2F_1\left(s+q-2,\frac{q}{2}-1,\frac{q}{2}+s,w^2\right) \\ &\quad {}_2F_1\left(p-\widehat{\Delta},\frac{p}{2},\frac{p}{2}+1-\widehat{\Delta},r^2\right) \mathrm{Disc}\mathcal{A}(r,w)\,. \end{aligned} \tag{3.84}$$

Now we want to change coordinates to z and $\bar{z}$. In the Lorentzian inversion formula (3.84), we integrate over the region of Lorentzian kinematics bounded by the curves $z=0$, $\bar{z}=1$ and $r^2 = z\bar{z} = 1\,, w>0$ - refer to fig. 3.3. Therefore in $(z,\bar{z})$ coordinates, the range of integration is $z=0$ to $z=1$ and $\bar{z}=1$ to $\bar{z}=\frac{1}{z}$. We use some hypergeometric identities to finally obtain the following inversion formula,

$$\begin{aligned} b(\widehat{\Delta},s) &= \int_0^1 \frac{dz}{2z} z^{-\frac{\widehat{\tau}}{2}} \int_1^{\frac{1}{z}} \frac{d\bar{z}}{2\pi\mathrm{i}} (1-z\bar{z})(\bar{z}-z)\bar{z}^{-\frac{\widehat{\Delta}+s}{2}-2}\, {}_2F_1\left(s+1,2-\frac{q}{2},\frac{q}{2}+s,\frac{z}{\bar{z}}\right) \\ &\quad {}_2F_1\left(1-\widehat{\Delta},1-\frac{p}{2},1+\frac{p}{2}-\widehat{\Delta},z\bar{z}\right) \mathrm{Disc}\,\mathcal{A}(z,\bar{z})\,. \end{aligned} \tag{3.85}$$

The cut between $w=0$ and $w=r$ has been mapped to the line $\bar{z}\in[1,1/z]$, and can be computed by going around the branch point at $\bar{z}=1$. Notice that, due to the inverse proportionality relation between w and $\bar{z}$ in eq. (3.35), $\mathrm{Disc}\,\mathcal{A}(r,w) = -\mathrm{Disc}\,\mathcal{A}(z,\bar{z})$. Eqs. (3.82) and (3.85) are analytic in s. However, we stress again that their validity cannot be established without knowledge of the behavior of $\mathcal{A}(r,w)$ for $w\to 0$, or equivalently $w\to\infty$. If $s_\star<\infty$, $s_\star$ being defined in eq. (3.67), the function $b(\widehat{\Delta},s)$ defined by eq. (3.82) is identical to the function obtained via the Euclidean inversion formula eq. (3.58) for all integer values of $s>s_\star$. But now, analyticity in s implies that the defect operators organize in analytic trajectories for $s>s_\star$.

Let us also note that, similarly to the Caron-Huot formula [74] discussed in sec. 3.1, the discontinuity in eq. (3.85) vanishes for a single defect block, and thus its validity cannot be verified term by term in a defect block decomposition. This is to be contrasted with the Euclidean formula (3.58), where the poles precisely arise order by order in the defect OPE expansion of the correlator, as we discussed around eq. (3.59).

Poles of $b(s,\widehat{\Delta})$ in $\widehat{\tau}$ arise from the lower bound of integration in z, and we can study eq. (3.85) in an expansion for small z,

$$\begin{aligned} b(\widehat{\Delta},s)\big|_{\text{poles}} &= \int_0^1 \frac{dz}{2z} z^{-\frac{\widehat{\tau}}{2}} \sum_{m=0} z^m \sum_{k=-m}^{m} c_{m,k}(\widehat{\Delta},s) B(z,\beta+2k)\,, \\ B(z,\beta) &:= \int_1^\infty \frac{d\bar{z}}{2\pi\mathrm{i}} \bar{z}^{-\frac{\beta}{2}-1} \mathrm{Disc}\,\mathcal{A}(z,\bar{z})\,, \end{aligned} \tag{3.86}$$

where $\beta=\widehat{\Delta}+s$, and where $c_{m,k}(\widehat{\Delta},s)$ are trivially obtained from the z expansion of the integrand in (3.85), with $c_{0,0}(\widehat{\Delta},s)=1$. Note that in eq. (3.86) we pushed the upper bound of the $\bar{z}$ integration to infinity, which will not modify the poles of $b(s,\widehat{\Delta})$ in $\widehat{\Delta}$, *provided*

$\mathcal{A}(z,\bar{z})$ behaves as (3.67). This follows from the behavior of (3.86) for small z and with $\bar{z} \sim \frac{1}{z}$. The upper bound of the $\bar{z}$ integration can only produce poles in s, and provided $g(z,\bar{z})$ grows as given in eq. (3.67) for $w \to 0$, then these poles will appear only for $s \leqslant s_\star$, that is for s outside the range of applicability of the formula. We shall come back to this point in the next section.

In a series expansion for small z, the functions $B(\beta, z)$ will give the following contributions to (3.86):

$$\sum_{m=0} z^m \sum_{k=-m}^{m} c_{m,k}(\widehat{\Delta}, s) B(z, \beta+2k) = \sum_m b_m(\widehat{\Delta}, s)\, z^{\frac{\widehat{\tau}_m(\beta)}{2}}\,, \tag{3.87}$$

with each term producing a pole for $\widehat{\tau} = \widehat{\tau}_m(\beta)$ in $b(\widehat{\Delta}, s)$, signaling a defect operator with that transverse twist. The OPE coefficients are obtained from the $\widehat{\Delta}$-residue of $b(s,\widehat{\Delta})$, at fixed s, according to (3.56), and so they are obtained from the coefficients in (3.87) after correcting by a Jacobian factor as

$$b^2_{s,\widehat{\Delta}} = \left(1 - \frac{d\widehat{\tau}_m(\beta)}{d\beta}\right)^{-1} b_m(\widehat{\Delta}, s)\Big|_{\beta=\widehat{\tau}_m(\beta)+2s}\,, \tag{3.88}$$

where $\widehat{\tau}_m(\beta)$ is the exact transverse twist of the m^{th} trajectory.

3.4.3 Contributions from a single bulk block

In general one does not have access to the full two-point function. However, the inversion (3.82) can be applied block by block in the bulk OPE decomposition of the correlator. Indeed, the bulk channel OPE still converges in the whole region $0 < z < 1/\bar{z} < 1$.

As discussed in section 3.3, knowledge of the low twist operators appearing in the bulk OPE translates into statements about the large transverse spin defect spectrum. The analysis of section 3.3 is not free from assumptions, similarly to the usual lightcone story applied to the four-point function of local operators. In the latter case, only recently have some of the assumptions started to be put on a firmer footing [204]. We can now recover the results of section 3.3 making use of the inversion formula. *Assuming* that the correlation function behaves as in eq. (3.67), the inversion formula shows that operators organize in analytic families for $s > s_\star$. Now we can prove that these trajectories have accumulation points for $s \to \infty$ at $\widehat{\tau} \to \Delta_\phi + 2m$. Furthermore, unlike in section 3.3, where we obtained the contribution of an exchanged bulk block to the defect spectrum in a $1/s$ expansion, the results obtained through (3.82) amount to the full contribution of the block at finite $s > s_\star$. These results therefore resum the $1/s$ expansion that could be obtained from carrying out the procedure of section 3.3 to all orders in $1/s$, as done in [29, 30, 33, 198] for the case without defects.

Let us see how the transverse derivatives come about in this context. For large transverse spin s, we see from (3.86) that the integral is dominated by $\bar{z} \to 1$. From the behavior of the bulk blocks in this limit (3.9), we find that the leading contributions come from operators with lowest twist, $\tau = \Delta - J$, which contribute as

$$\mathcal{A}(z,\bar{z}) \sim (1-\bar{z})^{-\Delta_\phi + \frac{\tau}{2}}\,(\text{a function of } z) + \dots\,, \qquad \text{for } \bar{z} \to 1\,. \tag{3.89}$$

The leading contribution is always the identity. As it is expected and we confirm below, the inversion of the identity yields the spectrum of the trivial defect, *i.e.*, the transverse derivatives and their OPE coefficients.

The only limitation in the above reasoning is that the integral in $\bar{z}$ that defines $B(z,\beta)$ in eq. (3.86) should be performed at finite z. However it will be convenient for practical reasons to work in a small z expansion, so we start by discussing under which circumstances this is allowed.

Small z expansion

The small z expansion does not commute with the infinite sum over bulk blocks. This is clear from the fact that while (3.86) should behave like (3.87), all the blocks except the identity contain a single logarithm of z as $z \to 0$, see eq. (3.24).One might complain that using eq. (3.24) of the block is not allowed here because the lightcone expansion of the block does not converge in the whole range $1 < \bar{z} < \infty$. One can instead use the expansion presented in section 4.2.1 of [64], where the expansion parameter is $\frac{1-\bar{z}}{\bar{z}}$, which can be integrated in desired range. The result still contains a single $\log z$.

This problem discussed plagues the Caron-Huot inversion formula 3.1 for the four-point function without defects as well and has been discussed in [74]. There, in section 4.3.2, a way out was found (see also [33]): after subtracting a known sum from the inversion formula, one can commute the small z limit with the block expansion. In this section, we will content ourselves with computing the contributions of individual bulk blocks after taking their small z limit, without any subtraction. We expect that the error we make becomes small when the defect spectrum differs from the one of the trivial defect by small anomalous dimensions: indeed, in this case also the r.h.s. of eq. (3.87) is well approximated by an expansion up to a single $\log z$ as $x^\alpha \approx 1 + \alpha \log x$ for small α.

This is what happens for instance at large spin, where the analytic functions in s that we find below resum part of the lightcone expansion. In some specific situations, the result is actually exact down to $s = s_\star$. At leading order in this coupling, a single logarithm of z is all that there is on the r.h.s. of eq. (3.87). Furthermore, in these examples a finite number of bulk channel blocks have a non-zero discontinuity, therefore we are free to take small z block by block. A similar situation was also discussed for the case of the four-point function in a theory without defects in [199]. It should be borne in mind that, to go beyond these results, a procedure similar to [74] is needed.

Transverse derivative operators: Exchange of the identity operator

As discussed above, the leading contributions to the large transverse spin spectrum come from leading twist bulk operators, and thus the identity operator, which has twist zero, dominates. All remaining operators are constrained by unitarity bounds to have $\tau = \Delta - J > 0$, provided $d > 2$ which is assumed throughout this work since we consider $q > 1$. If the identity is the only exchanged bulk operator then from (3.11) we find $s_\star = -\Delta_\phi$, and the inversion formula (3.82) is valid for all spins starting at $s = 0$. This happens for the case of the trivial (i.e.

no) defect, and thus we recover the full spectrum. However, if the identity is just part of a more complicated two-point function, the $w \to 0$ behavior, and thus $s_\star$ can be modified.

Taking the leading small z term in the identity contribution to $g(z,\bar{z})$ we find

$$B(z,\beta) = \int_1^\infty d\bar{z}\bar{z}^{-\frac{\beta}{2}-1} \frac{1}{2\pi \mathrm{i}} \mathrm{Disc}\left(\frac{(1-\bar{z})}{\sqrt{z\bar{z}}}\right)^{-\Delta_\phi} = \frac{z^{\frac{\Delta_\phi}{2}}\Gamma\left(\frac{\widehat{\Delta}+s+\Delta_\phi}{2}\right)}{\Gamma\left(\frac{\widehat{\Delta}+s-\Delta_\phi}{2}+1\right)\Gamma\left(\Delta_\phi\right)}, \tag{3.90}$$

where we see that in (3.86) this produces a pole in $\widehat{\tau} = \Delta_\phi$, corresponding to the leading twist defect primary operator. The residue of $B(z,\beta)$ matches precisely with the OPE coefficient of the trivial defect (3.13), for $m = 0$. Subleading powers in the small z expansion of the identity block and of (3.85) produce poles corresponding to the rest of the trivial spectrum, (3.12) with $m > 0$.

Since the bulk identity exchange corresponds to the leading contribution to the spectrum at large s, we thus recover the existence of transverse derivative operators with $\widehat{\tau} \to \Delta_\phi + 2m$ as $s \to \infty$. A main difference with respect to section 3.3 is that now we obtain the full OPE coefficient (3.13), instead of an asymptotic series in $1/s$.

Note that the integral in (3.90) naively diverges for large Δ_ϕ, but the result can be defined by analytic continuation and is finite, similarly to what was observed in [74]. The result of the Euclidean inversion formula gives a finite answer that is analytic in Δ_ϕ. For $\Delta_\phi < 1$ eq. (3.90) converges and thus it matches the result of the Euclidean inversion. The integral in (3.90) can then be analytically continued from there to $\Delta_\phi \geqslant 1$. This will also happen for the exchange of low dimensional bulk primaries, as the behavior for $\bar{z} \to 1$ of $\mathcal{A}(z,\bar{z})$ is controlled by the bulk channel OPE - see (3.89). For low dimensional bulk blocks then the result should also be obtained by analytic continuation.

Finally we note that $\mathrm{i}\,\mathrm{Disc}\mathcal{A}(z,\bar{z})$ in (3.85) does not have a definite sign, in contrast to the case of the double-discontinuity in [74]. This is clear from the identity contribution in (3.90) where

$$(2\mathrm{i})^{-1}\,\mathrm{Disc}\left((1-\bar{z})^{-\Delta_\phi}\right) = (\bar{z}-1)^{-\Delta_\phi}\sin(\pi\Delta_\phi)\,. \tag{3.91}$$

Even though positivity of the defect OPE coefficients requires the residues of $b(\widehat{\Delta},s)$ in (3.85) to have definite sign, as is the case for (3.90) above, this does not follow from the sign of the discontinuity.

Transverse derivative operators: Leading twist bulk contribution

The defect operator dimensions and OPE coefficients obtained from the inversion of the identity block will then be corrected for finite spin by the presence of all the remaining bulk blocks. We define the anomalous dimension of the transverse derivative operators whose dimensions approach $\Delta_\phi + 2m$ as

$$\gamma_{s,m} := \widehat{\tau}_m - (\Delta_\phi + 2m)\,. \tag{3.92}$$

As discussed above, if the $\gamma_{s,m}$ are small then we can consider the the small z limit of the bulk block decomposition.

We can draw from (3.89) a first general observation: if the exchanged operator has twist $\tau = 2\Delta_\phi + 2n$, with $n \geqslant 0$ an integer, the contribution of the relative block to the discontinuity of $\mathcal{A}(z,\bar{z})$ vanishes . In other words, exact double twist operators of the external operator in the bulk CFT contribute zero to the discontinuity in the two-point function and thus do not contribute under the inversion formula. Note that while the discontinuity naively vanishes also for negative integer n, the integral is divergent for $\bar{z} \to 1$ in this case. One must then first compute the discontinuity for arbitrary n and perform the integration. In the end, when n is taken to be a negative integer, the zero of the discontinuity cancels the divergence in the integral, and the final result of the inversion formula is finite. This is in precise agreement with the results of section 3.3.3: the bulk blocks with non vanishing discontinuity either give singular contributions to $g(z,\bar{z})$ as $\bar{z} \to 1$, or contributions that can be made singular by acting with the Casimir. The same behavior is observed for the inversion formula of the four-point function with no defects as pointed out in [74] – see also [207].

Let us rewrite the inversion formula in the following form:

$$\begin{aligned} b(\widehat{\Delta}, s) &= \int_0^1 \frac{dz}{2z} z^{-\frac{\widehat{\tau}}{2}} C(\beta, z)\,, \\ C(\beta, z) &= \int_1^\infty \frac{d\bar{z}}{2\pi \mathrm{i}} (1 - z\bar{z})(\bar{z} - z)\bar{z}^{-\frac{\beta}{2}-2}\, {}_2F_1\left(s+1, 2-\frac{q}{2}, \frac{q}{2}+s, \frac{z}{\bar{z}}\right) \\ &\quad {}_2F_1\left(1-\widehat{\Delta}, 1-\frac{p}{2}, 1+\frac{p}{2}-\widehat{\Delta}, z\bar{z}\right) \operatorname{Disc}\mathcal{A}(z,\bar{z})\,. \end{aligned} \tag{3.93}$$

Then we plug in the correlator with the contribution of the identity and a bulk primary O of twist τ and spin l to $C(\widehat{\Delta}, s, z)$:

$$\mathcal{A} = \left(\frac{(z\bar{z})^{1/2}}{(1-z)(1-\bar{z})}\right)^{\Delta_\phi} \left(1 + \lambda_{\phi\phi O}\, a_O\, g_{\Delta,l}(z,\bar{z})\right). \tag{3.94}$$

For the bulk block in eq. (3.94), we use the lightcone limit of the block in eq. (3.9). When expanded around $z=0$, this will also have logarithmic terms in z like in eq. (3.24). The terms of the form $(1-\bar{z})^{-\Delta_\phi + a}$ provide the discontinuity in the correlator which can be obtained as in eq. (3.91). With this input the generating function $C(\widehat{\Delta}, s, z)$ can be computed to be of the following form:

$$C(\beta, z) = A_I(\widehat{\Delta}, s, z) + \lambda_{\phi\phi O} a_O A_{\tau,l}(\widehat{\Delta}, s, z)\,, \tag{3.95}$$

$$A_I(\beta, z) = \sum_{m=0} I_m(\beta) z^{\frac{\Delta_\phi}{2}+m}\,, \tag{3.96}$$

$$A_{\tau,l}(\beta, z) = \sum_{m=0} \left(C_1^m(\beta) + C_2^m(\beta)\log z\right) z^{\frac{\Delta_\phi}{2}+m}\,. \tag{3.97}$$

$A_I(\beta,z)$ is given by the identity exchange, while $A_{\tau,l}(\beta,z)$ is given by the exchange of the other single block. $C_1^m(\beta)$ and $C_2^m(\beta)$ are expansion coefficients independent of $z, \bar{z}$. For example, for the leading transverse twist trajectory $m = 0$, these are given by,

$$C_1^0(\beta) = -\frac{2^{l+\tau}\Gamma\left(\frac{\beta+\Delta_\phi-\tau}{2}\right)\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)\left(\gamma_E+\psi\left(l+\frac{\tau}{2}\right)\right)}{\sqrt{\pi}\Gamma\left(\frac{\beta-\Delta_\phi}{2}+1\right)\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)}, \tag{3.98}$$

$$C_2^0(\beta) = -\frac{2^{l+\tau-1}\Gamma\left(\frac{\beta+\Delta_\phi-\tau}{2}\right)\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\beta-\Delta_\phi}{2}+1\right)\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)}, \tag{3.99}$$

where γ_E is the Euler-Mascheroni constant.

From (3.95), (3.96) and (3.97), we can deduce that the anomalous dimension of the m^{th} transverse twist trajectory is given as a function of s by:

$$\left.\frac{2\lambda_{\phi\phi O}a_O C_2^m}{I_m+\lambda_{\phi\phi O}a_O C_2^m}\right|_{\beta=\widehat{\tau}_m(\beta)+2s}. \tag{3.100}$$

The anomalous dimension when expanded in $\frac{1}{s}$ will have terms that are non-linear in the bulk channel OPE coefficient $\lambda_{\phi\phi O}a_O$. Since the lightcone bootstrap methods do not capture such corrections, we will drop these in order to compare our results to those from lightcone bootstrap. In eq. (3.100), the condition $\beta = \widehat{\tau}_m(\beta) + 2s$ can be implemented order by order in $\frac{1}{s}$, i.e. we can use the anomaolous dimension at j^{th} order to obtain that at the $(j+1)^{\text{th}}$ order in the $\frac{1}{s}$ expansion. However, this too shall give terms non-linear in the bulk channel OPE coefficient and hence we stick to setting $\widehat{\Delta} = \Delta_\phi + s + 2m$ to all orders in $\frac{1}{s}$. Thus we shall stick to calculating the anomalous dimensions as given by,

$$\gamma_{s,m} = \left.\frac{2\lambda_{\phi\phi O}a_O C_2^m}{I_m}\right|_{\widehat{\Delta}=\Delta_\phi+s+2m}. \tag{3.101}$$

In case I_m is zero the denominator should be the first non-zero order, this happens for instance if the external operator ϕ is perturbatively close to the unitarity bound, since in this case $b_{s,m}^2 = 0$ for $m \neq 0$ for $\Delta_\phi = \frac{d}{2} - 1$.

Similarly the OPE coefficients (3.13) are obtained from (3.95), (3.96) and (3.97) as follows:

$$b_{s,m}^2 = \left(1-\frac{d\gamma_{s,m}}{d\widehat{\Delta}}\right)^{-1}\left(I_m+\lambda_{\phi\phi O}a_O C_1^m\right)\Big|_{\beta=\widehat{\tau}(\beta)+2s}. \tag{3.102}$$

Once again, we wish to apply $\beta = \widehat{\tau}(\beta)+2s$ order by order in $\frac{1}{s}$ and retain only the terms that are at most linear in the bulk channel OPE coefficient $\lambda_{\phi\phi O}a_O$. Moreover, since a consistent truncation of $b_{s,m}^2$ at a given order in $\frac{1}{s}$ depends on the exact value of the twist τ of the bulk block, we shall present only the correction to the OPE coefficient due to the twist τ bulk block exchange over the exact value obtained from only the identity exchange. The correction to the OPE coefficient, to linear order in $\lambda_{\phi\phi O}a_O$, is thus given by:

$$\delta b_{s,m}^2 = \left(\frac{d\gamma_{s,m}}{d\widehat{\Delta}}I_m+\gamma_{s,m}\frac{dI_m}{d\widehat{\Delta}}+\lambda_{\phi\phi O}a_O C_1^m\right)\Big|_{\widehat{\Delta}=\Delta_\phi+s+2m}. \tag{3.103}$$

Applying these results to the bulk collinear block given in eq. (3.9) and expanding the answer for large s, we have recovered the results obtained with the lightcone approach of section 3.3, for different values of m and to second order in $\frac{1}{s}$. To this order, the Jacobian factor contribution in (3.102) is crucial. For the leading transverse twist trajectory, we present the results upto third order in $\frac{1}{s}$ here.

The anomalous dimension $\gamma_{s,m}$ is given by,

$$
\begin{aligned}
\gamma_{s,0} &= -\lambda_{\phi\phi O} a_O s^{-\frac{\tau}{2}} \left(\gamma_0^{(1)} + \frac{\gamma_0^{(2)}}{s} + \frac{\gamma_0^{(3)}}{s^2} + O\left(\frac{1}{s^3}\right)\right) + O\left(c^2_{\phi\phi O} a_O^2\right) , \qquad (3.104)\\
\gamma_0^{(1)} &= \frac{2^{l+\tau}\Gamma\left(\Delta_\phi\right)\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)}{\sqrt{\pi}\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)} ,\\
\gamma_0^{(2)} &= \frac{2^{l+\tau-3}\tau\left(\tau-4\Delta_\phi+2\right)\Gamma\left(\Delta_\phi\right)\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)}{\sqrt{\pi}\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)} ,\\
\gamma_0^{(3)} &= \frac{2^{l+\tau-7}\tau(2+\tau)\left(8-48\Delta_\phi+48\Delta_\phi^2+14\tau-24\Delta_\phi\tau+3\tau^2\right)\Gamma\left(\Delta_\phi\right)\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)}{3\sqrt{\pi}\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)} .
\end{aligned}
$$

The correction to the OPE coefficient is given by,

$$
\begin{aligned}
\delta b^2_{s,0} &= \lambda_{\phi\phi O} a_O s^{\Delta_\phi-\frac{\tau}{2}-1} \left(\lambda_0^{(1)} + \frac{\lambda_0^{(2)}}{s} + \frac{\lambda_0^{(3)}}{s^2} + O\left(\frac{1}{s^3}\right)\right) + O\left(c^2_{\phi\phi O} a_O^2\right) , \qquad (3.105)\\
\lambda_0^{(1)} &= -\frac{2^{l+\tau}\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)H_{-1+l+\frac{\tau}{2}}}{\sqrt{\pi}\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)} ,\\
\lambda_0^{(2)} &= \frac{2^{l+\tau-2}\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)\left(-2+(\tau-2\Delta_\phi)H_{-1+l+\frac{\tau}{2}}\right)}{\sqrt{\pi}\Gamma\left(\Delta_\phi-\frac{\tau}{2}-1\right)\Gamma\left(l+\frac{\tau}{2}\right)} ,\\
\lambda_0^{(3)} &= \frac{2^{l+\tau-7}\left(2\Delta_\phi-\tau-4\right)\left(2\Delta_\phi-\tau-2\right)\left(2\Delta_\phi-\tau\right)\Gamma\left(\frac{1}{2}+l+\frac{\tau}{2}\right)}{3\sqrt{\pi}\Gamma\left(\Delta_\phi-\frac{\tau}{2}\right)\Gamma\left(l+\frac{\tau}{2}\right)}\\
&\quad \left(-12+\left(2-6\Delta_\phi+3\tau\right)H_{-1+l+\frac{\tau}{2}}\right) ,
\end{aligned}
$$

H_x is a Harmonic number (extended to non-integers).

Note that the results in eq. (3.104) and eq. (3.105) are consistent with the results from lightcone bootstrap in sec. 3.3.3. We should emphasize again that the results presented above include only terms linear in the bulk OPE coefficient $\lambda_{\phi\phi O} a_O$ and so far we have not calculated non-linear corrections systematically.

This discussion also proves the existence of the individual transverse derivative operators, instead of the averaged statement obtained with the lightcone analysis. To be precise, for a given finite spin, it may happen that the contributions from the various bulk primaries to the residue of a certain pole sum up to zero. However, this cannot happen for sufficiently large spin, where the corrections from different exchanged operators are of different size. In this sense our results, similarly to those of [74], establish the existence of each individual transverse derivative operator for sufficient large s.

While the methods of section 3.3 only provide an asymptotic series in $\frac{1}{s}$, the inversion formula yields the contribution of a given bulk block to the anomalous dimension and OPE coefficient of a defect operator of any transverse spin $s > s_\star$. As an example, we shall now compute the exact anomalous dimension $\gamma_{s,0}$ of the leading transverse twist defect operator arising from the exchange of a bulk scalar.

Scalar operator exchange

Let us first obtain how a scalar operator O of dimension Δ contributes to $\gamma_{s,0}$. Apart from the scalar operator O, here we only take into account the contribution of the identity. While the bulk blocks are not known in closed form, we can make use of the representation of the scalar block as an infinite sum of hypergeometric functions as given in appendix B of [64]. Alternatively we could have used the recursion relation for the bulk blocks obtained in [64]. Taking the leading $z \to 0$ term of the block we apply (3.86) term by term in the block representation as an infinite sum. This amounts to a representation of the block as an infinite sum in powers of $\frac{\bar{z}-1}{\bar{z}}$ that converges for all of $\bar{z}$ in the integration region of (3.86). We then commute the integral over $\bar{z}$ with the infinite sum, and are able to resum the result to find

$$\gamma_{s,0}\Big|_{\Delta,l=0} = -c_{\phi\phi O}a_O \frac{\Gamma\left(\frac{\Delta+1}{2}\right)\Gamma(\Delta_\phi)\Gamma(s+1)\,{}_3F_2\left(\frac{\Delta-q+2}{2},\frac{\Delta}{2},\frac{\Delta-2\Delta_\phi+2}{2};\frac{\Delta}{2}+s+1,\Delta-\frac{d-2}{2};1\right)}{2^{-\Delta}\sqrt{\pi}\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\Delta_\phi-\frac{\Delta}{2}\right)\Gamma\left(\frac{\Delta}{2}+s+1\right)}\,. \tag{3.106}$$

We can proceed similarly for subleading transverse twists by keeping more terms in the small z expansion, but since the resulting anomalous dimension have longer expressions we do not display them here. Note that by taking the $z \to 0$ limit of the scalar block we are assuming small anomalous dimensions, and the result we present here should be seen as the leading contribution in the small parameter that controls the anomalous dimension.

Behavior of a single bulk block as $w \to 0$

While we cannot bound the growth of $g(z,\bar{z})$ as $w \to 0$, we can check the behavior of a single bulk block. This is trivial for the cases in which the blocks are known in closed form, and one finds that for codimension two and $d = 4, 6$, the behavior is power-law like. *Assuming* that the behavior of the bulk block is power-law like for all values of p and q we can solve both the quadratic and quartic Casimir equations in small w limit to find,

$$f_{\Delta,J} \sim w^{-\frac{p}{2}} f(r)\,, \qquad \text{as} \qquad w \to 0\,, \tag{3.107}$$

where $f(r)$ is a function of r that is fixed up to two constants by the Casimir equations. There is another nontrivial solution allowed by the Casimir that does not match the behavior obtained in codimension two. Putting in the behavior of the prefactor in (3.7) we find that for a single bulk block $s_\star = \frac{p}{2} - \Delta_\phi$. Of course this growth can be different for a theory where an infinite number of bulk blocks is exchanged.

3.5 Free theory with a defect

In this section, we shall consider a free theory with a defect which provides a simple instance where the inversion formula (3.85) does not converge down to zero transverse spin. Let us look at the defect spectra that can appear in the bulk-to-defect OPE of the free scalar ($\Delta_\phi = \frac{d-2}{2}$). It was shown in section B.1.1 of [64] that only two towers of defect operators are allowed by the equations of motion:

$$\begin{aligned} \widehat{\tau} &= \Delta_\phi\,, \\ \widehat{\tau} &= \Delta_\phi + 2 - q - 2s\,, \quad s \leqslant \frac{4-q}{2}\,. \end{aligned} \tag{3.108}$$

The first set is the tower of transverse derivatives, which are not allowed to acquire anomalous dimension. This agrees with the lightcone analysis, and with the inversion formula (3.85), since all the operators in the bulk OPE of ϕ with itself have zero discontinuity, except the identity. In turn, as remarked in subsection 3.4.3, the inversion of the identity precisely yields the spectrum of the trivial defect, with the OPE coefficients (3.13). Those vanish for $m > 0$ when Δ_ϕ is at the unitarity bound, and indeed only the leading transverse twist trajectory appears in eq. (3.108). What about the second tower in eq. (3.108)? These are isolated operators at low spin, as enforced by the unitarity bound in eq. (3.108). The lightcone expansion is blind to this kind of solutions. As we shall see now in a specific example, these operators also lie below the radius of convergence $s_\star$ of the inversion formula.

The simplest example of a non trivial defect in free theory is obtained by integrating a free field on a dimension $p = \frac{d}{2} - 1$ surface, which requires even $d \geqslant 4$ – see *e.g.* [64] for more details. It follows from (3.108) that the tower with bounded spin is only present if $d \leqslant 6$. In this case, a single defect operator with $s = 0$ and $\widehat{\Delta} = 0$ is allowed – the identity operator. The two-point function of the free field is given by

$$\langle \phi(1,1)\phi(z,\bar{z})\rangle = \frac{1}{((1-z)(1-\bar{z}))^{\Delta_\phi}} + \frac{a_\phi^2}{(z\bar{z})^{\Delta_\phi/2}}\,, \tag{3.109}$$

which indeed differs from that of a trivial defect (first addend in (3.109)) by the appearance of the defect identity (second addend in (3.109)). We now want to use the Lorentzian inversion formula (3.85) to recover the spectrum. We should check the behavior of $g(r,w)$ for $w \to 0$ (or similarly $w \to \infty$) before dropping the arcs near $w = 0$ and $w = \infty$ when going from (3.58) to (3.82). The two-point function has the following asymptotics:

$$g(r,w) = r^{\Delta_\phi} \left\langle \phi(1,1)\phi\left(rw, \frac{r}{w}\right)\right\rangle \sim a_\phi^2\, w^0 + \mathcal{O}(w^{\Delta_\phi})\,, \quad \text{for } w \to 0\,, \tag{3.110}$$

and so from (3.67) we find that the inversion formula (3.85) is valid for only for $s > s_\star = 0$. Indeed, while the inversion of the first addend in (3.109) reproduces the spectrum of the trivial defect, the second addend has zero discontinuity and does not contribute. Since the formula is not valid for $s = 0$, this is not at odds with the presence of the identity the defect OPE of ϕ.

3.6 The Ising twist defect

There exists a conformal defect with codimension two in the 3d Ising model, as supported by numerical evidence [37] and also by results from the epsilon expansion and the conformal bootstrap [56]. Local operators odd (even) under the $\mathbb{Z}_2$ flavor symmetry of the 3d Ising

CFT are multi-valued (single-valued) around the twist defect. As a consequence, the $\mathbb{Z}_2$ odd (even) defect spectrum takes half-integer (integer) values of the transverse spin s.

Following the literature, let us refer to the leading transverse twist primaries in the defect OPE of the spin field σ as ψ_s:

$$\sigma \sim \sum_s \psi_s + \text{higher } \widehat{\tau} , \qquad s \in \mathbb{N} + \frac{1}{2} . \tag{3.111}$$

The dimensions and OPE coefficients of the ψ_s have been calculated in the epsilon expansion in [56], *i.e.* by setting $d = 4 - \epsilon$ and keeping $q = 2$. To leading order in ϵ,

$$\widehat{\tau}_{\psi_s} = 1 - \left(\frac{1}{2} + \frac{1}{24s}\right)\epsilon + O(\epsilon^2) , \tag{3.112}$$

$$|b_{\sigma\psi_s}| = 1 + \frac{\psi(1) - \psi(s+1)}{4}\epsilon + O(\epsilon^2) , \qquad \psi(z) = \frac{d}{dz}\ln\Gamma(z) . \tag{3.113}$$

Let us interpret these values from the lightcone bootstrap point of view. The scaling dimension of σ is

$$\Delta_\sigma = 1 - \frac{\epsilon}{2} + O(\epsilon^2) , \tag{3.114}$$

so the ψ_s are easily identified as the leading trajectory of transverse derivative operators. The fusion $\sigma \times \sigma$, at the leading non-trivial order in ϵ, can be written as follows:

$$\sigma \times \sigma \sim 1 + \varepsilon + \{\tau = 2\Delta_\sigma\} + \text{higher twists} + O(\epsilon^2) , \tag{3.115}$$

where ε is the energy operator and $\{\tau = 2\Delta_\sigma\}$ denotes the conserved currents of the free theory, which do not acquire anomalous dimension at this order [208] – see also [29] and [209] for a more general understanding of this fact. The higher twist primaries are decoupled in the free theory, and so their OPE coefficient is $O(\epsilon)$. Hence, they also appear as operators with $\tau = 2\Delta_\sigma + 2m$ in this OPE. All together, the only primary contributing to the discontinuity is ε, which is therefore fully responsible for eqs. (3.112) and (3.113). The required OPE data were presented in [56]:

$$\Delta_\varepsilon = 2 - \frac{2\epsilon}{3} + O(\epsilon^2) , \qquad c_{\sigma\sigma\varepsilon}a_\varepsilon = -\frac{1}{8} + O(\epsilon) . \tag{3.116}$$

In fact, the full result (3.112), (3.113) is encoded in the leading transverse spin correction. Indeed, plugging eq. (3.116) in eq. (3.31), we reproduce the value of the anomalous dimension $\gamma_{s,0} = -\frac{1}{24s}\epsilon$. Furthermore, the correction $b_{\min}$ in eq. (3.33) is $O(\epsilon^2)$, and indeed the square root of eq. (3.13) reduces to (3.113). Despite the simplicity of the result, it is not obvious why the large s expansion of the anomalous dimension should truncate at order $1/s$. We can address the question by means of the inversion formula. By evaluating the single block contribution eq. (3.106) with $q = 2$, $d = 4 - \epsilon$ and the CFT data in eqs. (3.114) and (3.116), we indeed get

$$\gamma_{s,0} = -\frac{\epsilon}{24}\frac{1}{s+1}\,{}_2F_1(1,1,s+2,1) + O(\epsilon^2) = -\frac{\epsilon}{24s} + O(\epsilon^2) , \tag{3.117}$$

where the last manipulation is valid for $s > 0$. It is interesting to notice that each bulk collinear block contributes an infinite series in $1/s$, and the final result emerges from infinitely many exact cancellations. We were not able to find the contribution of a single scalar block to the OPE coefficient in closed form. However, from the computation of the OPE coefficient as an infinite sum, discussed in section 3.4.3, we can easily check that the same cancellations are in place: this time, after appropriately including the Jacobian in eq. (3.88), no contribution is left at order ϵ. Therefore, we also recover eq. (3.113). We can also predict the existence of the higher transverse twist primaries, with $\widehat{\tau} \to \Delta_\sigma + 2m$ at large spin, whose OPE coefficients, for $m \neq 0$, are of order ϵ as clear from the fact that (3.13) vanishes for $m > 0$ and Δ_ϕ at the unitarity bound.

Let us conclude with some comments on the $\mathbb{Z}_2$ even defect spectrum. In free theory, the leading transverse twist operators are bilinear of the ψ_s, and all operators $\psi_j \psi_{s-j}$ with integer transverse spin s have the same transverse twist $\widehat{\tau} = 2-\epsilon$. This $\lfloor \frac{s+1}{2} \rfloor$-fold degeneracy is lifted at the Wilson-Fisher fixed point, and we parametrize the eigenvalues of the matrix of anomalous dimensions as follows:

$$\widehat{\tau}_{s,j} = 2 + \epsilon\,(\delta_{s,j} - 1) + O(\epsilon^2)\,. \tag{3.118}$$

In [56], it was pointed out that the following accumulation points exist at infinite transverse spin:

$$\delta_{\infty,j} = -\frac{1}{12(2j-1)}\,, \qquad j = 1, 2, \ldots\,. \tag{3.119}$$

The results of section 3.3 predict an additional accumulation point: the leading transverse derivative of the energy operator ε, that is,

$$\delta_{\infty,0} = \frac{1}{3}\,. \tag{3.120}$$

In fact it can be proved independently that this accumulation point exists and that furthermore eqs. (3.119) and (3.120) comprise all the anomalous dimensions of this class of operators. As for the accumulation points (3.119), those are not transverse derivatives of ε, and therefore we should expect their OPE coefficient to be subleading at large s. We did not check this fact. Both the lightcone bootstrap and the use of the inversion formula are complicated by the presence of infinitely many bulk blocks contributing already at order ϵ, so we leave this analysis for future work.

Chapter 4

Mellin representation of fermionic correlation functions

In the previous chapter, we discussed two powerful analytical tools, namely the lightcone bootstrap and the Lorentzian OPE inversion formula, in the context of CFTs with defects. These methods revealed to us the phenomenon of universality at large transverse spin in defect CFTs and also the analyticity of OPE data as a function of transverse spin above a certain threshold and also enabled us to calculate corrections to this universal behavior thus giving us access to non-trivial dynamical information in interacting CFTs. Both the lightcone bootstrap and the Lorentzian OPE inversion formula are methods that employ the position space representation of correlation functions.

As motivated previously in chap. 1, the Mellin representation of conformal correlation functions [79, 80] offers a particularly insightful and practically useful approach to studying CFTs much like the momentum space representation for massive QFTs. A Mellin amplitude encodes OPE data in its poles and residues respectively and factorizes onto lower point amplitudes. There are accumulation points populated by double trace like operators in the spectrum of a CFT in dimensions higher than two (as discussed in sec. 3.1) and hence the Mellin amplitude is not a meromorphic function in general. However, for conformal large N gauge theories, the Mellin amplitude as defined by Mack [79] accounts for only the single trace operators and is therefore a meromorphic function.

The present literature on Mellin amplitudes focuses almost exclusively on correlation functions of scalar operators. It is natural to ask if one can define Mellin amplitudes for correlation functions of operators with spin and make similar progress conceptually and with applications as for scalar correlation functions. Mellin amplitudes for correlation functions of scalars and one integer spin operator were defined in [85] with the purpose of studying factorization of scalar Mellin amplitudes onto lower point Mellin amplitudes. However the Mellin amplitudes of the spinning correlators themselves have not been studied as such. This deficiency is especially significant in the context of fermionic conformal correlation functions. Fermionic operators do not appear in the OPE of scalar operators. Therefore, if one desires to access the fermionic sector of a CFT, it is necessary to consider correlation functions with at least spin-half operators. Moreover, spinning correlation functions in general can potentially provide us with more information on CFT data than scalar correlation functions.

In this chapter, we shall address this shortcoming by defining and studying Mellin amplitudes for correlation functions with spin-half fermions. We define Mellin amplitudes for the four-point function of two fermions and two scalars and the four-point function of four fermions. For simplicity, we restrict our analysis of the analyticity properties of the Mellin amplitude to three dimensions. Defining the Mellin amplitude involves making a choice of a basis of tensor structures and the Mellin amplitude has one component corresponding to each basis element. Generically, the separation of the tensorial part might introduce spurious singularities in the conformal blocks, as noted in [164, 168]. Therefore not all bases are suitable for defining a Mellin amplitude with the desired analyticity properties. After defining the Mellin amplitude suitably, we proceed to examine the pole structure by looking at the behavior of the correlator in the OPE limit. This also makes the factorization of the four-point Mellin amplitude manifest.

The three-point function of two fermions and a boson has multiple tensor structures. Generically, this results in each component of the Mellin amplitude having more than one distinct series (two in our case) of poles corresponding to each primary operator exchanged in the OPE in a given channel. We always choose tensor structures of definite parity for both the three-point and four-point functions as this choice leads to simplifications in the pole structure when the three-point functions are of definite parity. It must be noted that the pole structure of the Mellin amplitude is related to the choice of basis and is tunable as such. After this preliminary analysis of the properties of the Mellin amplitude, we shall compute some Mellin amplitudes corresponding to tree level Witten diagrams and tree level conformal Feynman integrals. These examples illustrate the generic predictions on the pole structure considering the parity of the three-point functions in each case. The definition also trivially extends to n-point functions when supplemented with a concrete choice of tensor structures.

In sec. 4.1, we start with a basic review of Mellin amplitudes for scalar correlators and for correlators with one integer spin operator and scalar operators. We shall then discuss the basis of tensor structures that we would be using in each case in sec. 4.2. Thereafter, we shall define the Mellin amplitude for fermionic correlation functions in sec. 4.3 and present the pole structure of the fermion scalar four-point correlator and the four fermion correlator in sec. 4.4 and sec. 4.5 respectively. In sec. 4.6, we shall see some results for Mellin amplitudes corresponding to tree level Witten diagrams and in sec. 4.7, we shall move on to Mellin amplitudes for conformal Feynman integrals.

This chapter is based on the author's publication [2] and contains excerpts from the same. Unless otherwise mentioned, we shall be using the embedding space notation discussed in chap. 2.

4.1 Inspiration

Let us begin by briefly reviewing the basics of Mellin amplitudes for scalar correlators and appreciate why the Mellin representation is a natural one for conformal correlators. The

Mellin amplitude for the connected part of a scalar correlator was defined by Mack [79] in the following manner (Euclidean signature):

$$\langle O_1(X_1)O_2(X_2)\cdots O_n(X_n)\rangle_c = \int [ds_{ij}] \prod_{i<j} \Gamma(s_{ij})\, X_{ij}^{-s_{ij}} \mathcal{M}(\{s_{ij}\})\,. \tag{4.1}$$

The integral in eq. (4.1) is a Mellin-Barnes integral[1] and the contours run parallel to the imaginary axis. The Mellin variables s_{ij} are not all independent but satisfy the following constraints,

$$\Delta_i - \sum_{j\neq i} s_{ij} = 0 \quad \forall i \tag{4.2}$$

The measure $[ds_{ij}]$ includes a set of independent Mellin variables and an overall factor of $\left(\frac{1}{2\pi i}\right)^{\frac{n(n-3)}{2}}$. These conformality constraints in eq. (4.2) ensure that the right hand side of eq. (4.1) transforms properly under conformal transformations. The number of independent Mellin variables s_{ij} is $\frac{n(n-3)}{2}$ which is the same as the number of independent cross-ratios. For $n > d+2$, the dimension of the conformal moduli space is less than this (see [168]) and the associated Mellin amplitude is non-unique (see [159]).

The conformality constraints can be interpreted in terms of Mellin momenta k_i with $k_i \cdot k_j = s_{ij}$ and an on-shell condition $k_i^2 = -\Delta_i$ as the overall conservation of Mellin momentum $\sum_i k_i = 0$. One can thus relate the Mellin variables to Mandelstam variables $S_{i_1\cdots i_a}$ as

$$S_{i_1\cdots i_a} = -\left(k_{i_1} + \cdots + k_{i_a}\right)^2 = -2\sum_{l<k\leq a} s_{i_l i_k} + \sum_{j=1}^{a} \Delta_{i_j}\,. \tag{4.3}$$

The location of the poles in a given Mandelstam variable $S_{i_1 i_2}$ is at the twists of the operators in the OPE of $O_{i_1}O_{i_2}$ that contribute to the correlator. The Mellin amplitude factorizes at these poles and the residue is proportional to the Mellin amplitudes of the corresponding lower point correlators as dictated by the OPE.

As an example, let us look at the case of the four-point function.

$$\begin{aligned}\langle O_1(X_1)O_2(X_2)O_3(X_3)O_4(X_4)\rangle_c &= \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}} \frac{\mathcal{A}(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}}(X_{34})^{\frac{\Delta_3+\Delta_4}{2}}}\,,\\ u &= \frac{X_{12}X_{34}}{X_{13}X_{24}}, \qquad v = \frac{X_{14}X_{23}}{X_{13}X_{24}},\end{aligned} \tag{4.4}$$

The conformal amplitude $\mathcal{A}(u,v)$ can now be expressed in the Mellin representation in the following manner:

$$\begin{aligned}\mathcal{A}(u,v) &= \int_{c_s-i\infty}^{c_s+i\infty} \frac{ds}{4\pi i} \int_{c_t-i\infty}^{c_t+i\infty} \frac{dt}{4\pi i}\, \mathcal{M}(s,t)\, u^{\frac{s}{2}}\, v^{-\frac{s+t-\Delta_1-\Delta_4}{2}}\, \Gamma\left(\frac{\Delta_1+\Delta_2-s}{2}\right)\\ &\quad \Gamma\left(\frac{\Delta_3+\Delta_4-s}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-t}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_4-t}{2}\right)\\ &\quad \Gamma\left(\frac{s+t-\Delta_1-\Delta_4}{2}\right)\Gamma\left(\frac{s+t-\Delta_2+\Delta_3}{2}\right)\,.\end{aligned} \tag{4.5}$$

[1] See appendix. C for a brief review of the Mellin transform, Mellin-Barnes integrals and a Mellin space representation of the delta function.

In eq. (4.5), the Mellin variables s_{ij} have been traded for the Mandelstam variables s and t.

$$s = -(k_1+k_2)^2 = \Delta_1+\Delta_2-2s_{12}, \qquad t = -(k_1+k_3)^2 = \Delta_1+\Delta_3-2s_{13}. \tag{4.6}$$

Mack [79] realized that for every conformal primary with twist τ contributing to the conformal block expansion of $\mathcal{A}(u,v)$ in the direct channel, $\mathcal{M}(s,t)$ has poles at $s=\tau+2m$, $m=0,1,2,\cdots$ where $m=0$ corresponds to the primary and the leading twist descendants (and similarly for the other channels).

The factors of gamma functions in eq. (4.6) also contribute poles, for example at $s=\Delta_1+\Delta_2+2m$. These poles correspond to operators of the form $O_1\partial^l\left(\partial^2\right)^m O_2$ that contribute to the conformal block expansion and have the said values of twist when the anomalous dimensions are suppressed. In large N gauge theories, these are the familiar double trace operators. The Mellin amplitude then accounts for the contributions from only single trace operators and is a meromorphic function of the Mellin variables.

Owing to its pole structure, the Mellin amplitude in eq. (4.5) can be expressed in the following form,

$$\mathcal{M}(s,t)=\sum_{\mathcal{O}}\sum_{m=0}^{\infty}\frac{\lambda_{12\mathcal{O}}\,\lambda_{\mathcal{O}34}\,\mathcal{Q}_{l,m}(t)}{s-\tau-2m}+\text{regular terms}\,. \tag{4.7}$$

The sum in eq. (4.7) are over operators $\mathcal{O}$ (with twist τ and spin l) exchanged in the block expansion of $\mathcal{A}(u,v)$. $\mathcal{Q}_{l,m}(t)$ is a polynomial of degree l (for all m) in the variable t. These polynomials satisfy a Sturm-Liouville type finite difference equation [82, 83] that can be obtained by acting with the quadratic conformal Casimir operator on the conformal blocks expressed in a Mellin representation.

Eq. (4.7) immediately makes the OPE data in the correlator $\mathcal{A}(u,v)$ manifest in its poles and residues and is akin to the Källén-Lehmann spectral representation. We also see that the residue at a given pole of the four-point Mellin $\mathcal{M}(s,t)$ is proportional to a product of two OPE coefficients $\lambda_{12\mathcal{O}}\lambda_{\mathcal{O}34}$. On the other hand, from eq. (4.1), we know that the Mellin amplitude for the three-point function $\langle O_1O_2O\rangle$ is $\lambda_{12\mathcal{O}}$ (upto normalization). This is an explicit manifestation of the factorization of Mellin amplitude onto lower point Mellin amplitudes.

Now we can ask ourselves how to define the Mellin amplitude for a correlator of spinning operators. Although there does not seem to be any canonical answer to this problem, we take inspiration from the approach adopted in [85]. A spinning conformal correlator naturally has tensor structures associated with it that carry the Lorentz indices. For any given correlator, there is generically more than one independent tensor structure, and with each tensor structure we have an associated function of the cross-ratios, say $\mathcal{A}_i(u,v)$ (the index i runs over the possible independent tensor structures). Just like in eq. (4.1), we can express each $\mathcal{A}_i(u,v)$ in the Mellin representation to obtain $\mathcal{M}_i(s,t)$. For a given choice of tensor structures, the set $\{\mathcal{M}_i(s,t)\}$ uniquely defines the Mellin amplitude associated to the spinning correlator.

In the following sections, we shall begin with a discussion on tensor structures for fermionic correlators and this will mostly involve reviewing and using results from the literature. Then we shall define the Mellin amplitudes concretely for the fermion four-point function and fermion-scalar four-point function and establish their pole structure and factorization properties.

The computational advantages that comes with the Mellin representation has been demonstrated quite extensively in the context of Witten diagrams [86–91], exact holographic correlators [92, 93] (see also [94, 95]) and conformal Feynman integrals [97–99]. Follow in their footsteps, we shall compute some tree level Witten diagrams and some conformal Feynman integrals corresponding to fermionic conformal correlators. This shall also illustrate some of the general properties of such Mellin amplitudes in regimes controlled by small parameters $\frac{1}{N}$ and the coupling constant respectively.

4.2 Tensor structures

In order to discuss a Mellin representation for fermionic conformal correlators, first we have to discuss the tensor structures that appear in these correlators and select a basis for each. As mentioned before, we shall restrict the discussion to the case of 3d Minkowski spacetime (signature $-++$) for simplicity and also assume that all operators of the same spin have different conformal dimensions. We shall also assume that all the operators have different conformal dimensions. We shall be using the embedding formalism for spinors developed in [161, 164] that has been discussed in sec. 2.3.4.

There does not seem to be any canonical choice for the basis of tensor structures. We shall choose basis elements of definite parity. One should also note that not every choice of basis is suitable for defining the Mellin amplitude such that the poles of the amplitude can be associated with operators contributing to the conformal block expansion of the correlator. This is because for certain choices of bases, as explained in Section 4.4 of [168], there maybe spurious singularities in the conformal blocks. For example, in the context of the fermion four-point function, the naive conformal blocks associated with the basis in Section 2.4 of [164] have singularities at $z = \bar{z}$. A neat way to count the number of independent tensor structures and to find relations between tensor structures (when it is otherwise tedious to do so) is to shift to a conformal frame [168]. We shall begin with a quick review of the general principle and the relevant results and then move on to stating our choice of bases for the relevant three-point and four-point functions. We stick to the choice of bases made in [164, 179] making an independent choice of basis only for the fermion four-point function.

Let us now describe the conventions followed for defining the tensor structures in embedding space. Tensor structures are fixed by the 5d Lorentz invariance, transversality and homogeneity conditions on the embedding space operators that we just discussed. Although the homogeneity condition on the entire correlator is fixed by the conformal symmetry, the homogeneity on the tensor structures themselves depends on the chosen normalization and thus is a matter of choice. Now, we shall state our convention for this normalization of tensor structures that we shall follow throughout this chapter.

From the brief discussion on the embedding space formalism we know that the embedding space operators (fermions and bosons with any value of spin l) satisfy the following homogeneity property respectively,

$$\mathcal{O}(aX, bS) = a^{-\Delta-l} b^{2l}\, \mathcal{O}(X, S)\,. \tag{4.8}$$

Consequently, an n-point function of operators with dimension Δ_i and spin l_i satisfies the following homogeneity property,

$$\langle \mathcal{O}_1\left(a_1X_1, b_1S_1\right) \cdots \mathcal{O}_n\left(a_nX_n, b_nS_n\right)\rangle = \prod_{i=1}^{n} a_i^{-\Delta_i} \left(\frac{b_i^{2l_i}}{a_i^{l_i}}\right) \langle \mathcal{O}_1\left(X_1, S_1\right) \cdots \mathcal{O}_n\left(X_n, S_n\right)\rangle\ . \tag{4.9}$$

For example, an n-point function of $2K$ spin half fermions and M scalars ($2K + M = n$), which shall be most relevant for us, should satisfy the following homogeneity property,

$$\begin{aligned}&\langle \Psi_1\left(a_1X_1, b_1S_1\right) \cdots \Psi_{2K}\left(a_{2K}X_{2K}, b_{2K}S_{2K}\right) \Phi_{2K+1}\left(a_{2K+1}X_{2K+1}\right) \cdots \Phi_n\left(a_nX_n\right)\rangle \\ &= \prod_{i=1}^{n} a_i^{-\Delta_i} \prod_{j=1}^{2K} \frac{b_j}{\sqrt{a_j}} \langle \Psi_1\left(X_1, S_1\right) \cdots \Psi_{2K}\left(X_{2K}, S_{2K}\right) \Phi_{2K+1}\left(X_{2K+1}\right) \cdots \Phi_n\left(X_n\right)\rangle\ .\end{aligned} \tag{4.10}$$

We define the tensor structures $\mathbb{T}_k$ for a generic spinning correlator such that they entirely account for the factor $\prod_{i=1}^{n} \frac{b_i^{2l_i}}{a_i^{l_i}}$ in the homogeneity relation eq. (4.9). In other words, $\mathbb{T}_k$ are defined such that they satisfy the following homogeneity relation,

$$\mathbb{T}_k\left(b_1S_1, \cdots, b_nS_n;\, a_1X_1, \cdots, a_nX_n\right) = \prod_{i=1}^{n} \frac{b_i^{2l_i}}{a_i^{l_i}} \mathbb{T}_k\left(S_1, \cdots, S_n;\, X_1, \cdots, X_n\right)\ . \tag{4.11}$$

To make it more concrete, let us now see explicitly how the three-point functions and four-point functions of interest look like with the chosen conventions for the tensor structures. The three-point function of two fermions and one bosonic operator is of the following form,

$$\langle \Psi_1\left(X_1, S_1\right) \Psi_2\left(X_2, S_2\right) \Phi(X_3, S_3)\rangle = \sum_k \frac{\mathbb{T}_k \lambda_{123,k}}{X_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} X_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}} X_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}}\,, \tag{4.12}$$

where $\lambda_{123,k}$ are the structure constants for this three-point function. k runs over all the independent tensor structures.

Similarly, any four-point function can be expressed in the following form,

$$\langle \mathcal{O}_1\left(X_1, S_1\right) \cdots \mathcal{O}_4\left(X_4, S_4\right)\rangle \;=\; \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}} \frac{\sum_k \mathbb{T}_k \mathcal{A}_k(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}} (X_{34})^{\frac{\Delta_3+\Delta_4}{2}}}\,, \tag{4.13}$$

where $\mathcal{A}_k(u, v)$ are functions of cross-ratios and contain dynamical information.

It may seem surprising that both eq. (4.12) and eq. (4.13) lack any manifest dependence on the spin of the operators. This is because that spin dependence is entirely captured by the tensor structures $\mathbb{T}_k$ which one has to choose suitably in each case. This should be clear from eq. (4.9) and eq. (4.11) wherein we have normalized the tensor structures to entirely account for the factor $\prod_{i=1}^{n} \frac{b_i^{2l_i}}{a_i^{l_i}}$. Therefore as per our definition, the non-tensorial part of the correlator only accounts for the factor $\prod_i a_i^{-\Delta_i}$ in eq. (4.9) which is independent of the spin of the operators.

We should also note that different choices of normalization for any tensor structure can result in a difference to the corresponding Mellin amplitude only if they are different by factors of invariants (otherwise the difference has to be accounted for with a modification in the definition of the Mellin amplitude). When such an invariant is a product of cross-ratios, it would cause a simple shift in the poles of the Mellin amplitude.

The following discussion that will be focussing on three dimensions can also be easily generalized. The nature of spinors changes with dimension and signature. 4d tensor structures in the embedding formalism and blocks have been discussed in [163, 210]. The relevant setup is coherently presented in [211] and the setup is implemented in the freely available Mathematica package "CFTs4d". One can easily use this Mathematica package to obtain independent tensor structures (with expressions in both embedding space and the conformal frame) for upto four-point functions for any kind of correlator. In three dimensions, all operators exchanged in the OPEs can be taken to be symmetric representations of the double cover of the Lorentz group. However, in higher dimensions, one has to also consider mixed symmetry representations (see [181]).

4.2.1 Counting tensor structures

Correlators expressed in embedding space variables are manifestly covariant with conformal transformations and are easy to work with. However the downside is that there is a great deal of redundancy in all the possible tensor structures one can write. Sometimes it is easy to see relations between different tensor structures through gamma matrix commutation relations or simple Fierz identities, but in general this is a tedious matter. Let us now quickly review an elegant and neat method to count the number of independent tensor structures and figure out the web of relations between the different tensor structures in embedding space put forward in [168]. The key idea is to go to a suitable conformal frame by Lorentz transformations as depicted elegantly in [168]. In this paper, they prove that independent tensor structures in a n-point function are in one-to-one correspondence with the singlets (scalars for parity even tensor structures and pseudo-scalars for parity odd tensor structures) of the little group that leaves the configuration of points (at which operators in the correlator are inserted) in this conformal frame invariant. These singlets can be represented by

$$\mathrm{Res}^{O(d)}_{O(d+2-m)} \bigotimes_{i=1}^{n} \rho_i \,, \qquad m = \mathrm{Min}\{n, d+2\}\,. \tag{4.14}$$

Res^G_H denotes the restriction of a representation of G to a representation of $H \subseteq G$. ρ_i is the representation of the Lorentz group in which the operator at the i^{th} position in the correlator transforms. If parity is not a symmetry of the theory, then we should replace $O(\cdot)$ with $SO(\cdot)$. To consider half-integer spin representations one has to use the double cover of $SO(\cdot)$ which is $Spin(\cdot)$ and for parity symmetric theory one has to make a choice of the

$Pin(\cdot)$ group. If $n \geq d+2$, one can form a parity odd invariant and consequently restrict to using only parity even tensor structures. The following is a parity odd invariant suggested in [168],

$$w = \frac{\epsilon_{\mu_1 \cdots \mu_{d+2}} X_1^{\mu_1} \cdots X_{d+2}^{\mu_{d+2}}}{\sqrt{X_{12} X_{23} \cdots X_{d+1,d+2} X_{d+2,1}}} \,. \tag{4.15}$$

If there are identical operators in the correlator, permutation symmetries result in further reductions in the number of independent tensor structures as explained concretely in [168], but we shall stick to assuming operators with different dimensions.

A conformal frame for n points is any fixed configuration of points to which one can always map any n points using conformal transformations. The most familiar example of this is probably the conformal frame where four points are mapped to 0, 1 (along any axis x), ∞ and $(z, \bar{z})$ (on a chosen plane containing the axis x). In general, relations between embedding space tensor structures can be obtained by choosing a conformal frame and expressing them in terms of the conformal frame tensor structures which are free of redundancies, and then simple linear algebra gives relations between the different embedding space tensor structures.

The counting of tensor structures for 3d fermions has already been done in Section 4.2 of [168]. We quote the relevant results here. The number of independent tensor structures (of definite parity, indicated by the signs in the superscript) for the three-point function of operators with spins l_i is given by,

$$\begin{aligned} N_{3d}^{\pm} &= \frac{N_{3d}\,(l_1, l_2, l_3) \pm \kappa}{2} \,, \\ N_{3d} &= (2l_1+1)(2l_2+1) - p(p+1) \,, \qquad\qquad p = \mathrm{Max}\,(l_1 + l_2 - l_3, 0) \quad l_1 \leq l_2 \leq l_3 \,. \end{aligned} \tag{4.16}$$

$\kappa = 1$ when all the operators are of integer spin and $\kappa = 0$ otherwise.

The number of independent n-point tensor structures for $n \geq 4$ [168] is given by,

$$N_{3d}\,(l_1, l_2, l_3, l_4) = \prod_{i=1}^{n} (2l_i + 1) \,. \tag{4.17}$$

If there is at least one half-integer spin operator, we can take an equal number of parity odd structures and parity even structures. Thus one can choose two parity even and two parity odd tensor structures for $\langle \psi_1 \psi_2 O_3 O_4 \rangle$.

4.2.2 Three-point functions

Let us now state the tensor structures for the relevant three-point functions which are those of two spin half fermions ψ_1, ψ_2 and a bosonic operator $O_{3,l}$ and of one spin half fermion ψ_1 one scalar O_2 and one fermionic operator of any spin $\psi_{3,l}$. The structures for the three-point function of two fermions and a scalar $\langle \psi_1 \psi_2 O_3 \rangle$ can be taken to be,

$$r_{di}^{+} = \frac{\langle S_1 S_2 \rangle}{\sqrt{X_{12}}} \to \frac{\not{x}_{12}}{|x_{12}|}, \qquad\qquad r_{di}^{-} = \frac{\langle S_1 X_3 S_2 \rangle}{\sqrt{X_{13} X_{32}}} \to \frac{\not{x}_{13} \not{x}_{32}}{|x_{13}||x_{23}|}. \tag{4.18}$$

For the three-point function of two fermions and a spin l bosonic operator $\langle\psi_1\psi_2 O_{3,l}\rangle$ $(l>0)$, we have two parity even structures and two parity odd ones which can taken to be,

$$\begin{aligned}
r^+_{di,1} &= \frac{\langle S_1S_2\rangle\langle S_3X_1X_2S_3\rangle^l}{X_{12}^{\frac{l+1}{2}}X_{13}^{\frac{l}{2}}X_{23}^{\frac{l}{2}}}, \qquad r^+_{di,2} = \frac{\langle S_1S_3\rangle\langle S_2S_3\rangle\langle S_3X_1X_2S_3\rangle^{l-1}}{X_{12}^{\frac{l-1}{2}}X_{13}^{\frac{l}{2}}X_{23}^{\frac{l}{2}}}, \\
r^-_{di,3} &= \frac{\langle S_3X_1X_2S_3\rangle^{l-1}}{X_{12}^{\frac{l}{2}}X_{13}^{\frac{l+1}{2}}X_{23}^{\frac{l+1}{2}}}\left[X_{23}\langle S_1S_3\rangle\langle S_2X_1S_3\rangle + X_{13}\langle S_2S_3\rangle\langle S_1X_2S_3\rangle\right], \\
r^-_{di,4} &= \frac{\langle S_3X_1X_2S_3\rangle^{l-1}}{X_{12}^{\frac{l}{2}}X_{13}^{\frac{l+1}{2}}X_{23}^{\frac{l+1}{2}}}\left[X_{23}\langle S_1S_3\rangle\langle S_2X_1S_3\rangle - X_{13}\langle S_2S_3\rangle\langle S_1X_2S_3\rangle\right].
\end{aligned} \tag{4.19}$$

For $l=0$, $r^+_{di,1}$ goes to r^+_{di} and $r^-_{di,3}$ goes to r^-_{di}.

The 3d expressions can be obtained from the rule :

$$\langle S_1X_2X_3\cdots X_{k-1}S_k\rangle \to \not{x}_{12}\not{x}_{23}\cdots\not{x}_{k-1,k}\not{x}_{k-1,k}\,. \tag{4.20}$$

We shall denote the 3d expressions corresponding to products of the form $\langle S_iX_a\cdots X_bS_m\rangle\langle S_kX_u\cdots X_vS_l\rangle$ as $\left[\not{x}_{ia}\cdots\not{x}_{bm}\right]\left[\not{x}_{ku}\cdots\not{x}_{vl}\right]$.

For the three-point function of one spin half fermion one scalar and one fermionic operator of any spin $\langle\psi_1O_2\psi_{3,l}\rangle$ $(l>\frac{1}{2})$, we can take the following tensor structures,

$$\begin{aligned}
r^+_{cr} &= \frac{\langle S_1S_3\rangle\langle S_3X_1X_2S_3\rangle^{l-\frac{1}{2}}}{X_{12}^{\frac{l}{2}-\frac{1}{4}}X_{13}^{\frac{l}{2}+\frac{1}{4}}X_{23}^{\frac{l}{2}-\frac{1}{4}}}, \\
r^-_{cr} &= \frac{\langle S_1X_2S_3\rangle\langle S_3X_1X_2S_3\rangle^{l-\frac{1}{2}}}{X_{12}^{\frac{l}{2}+\frac{1}{4}}X_{13}^{\frac{l}{2}-\frac{1}{4}}X_{23}^{\frac{l}{2}+\frac{1}{4}}}.
\end{aligned} \tag{4.21}$$

Note that we have chosen the same tensor structures for the three-point functions as in [164, 179] only with different normalization.

4.2.3 Mixed fermion-scalar four-point function

Let us now consider correlators with two fermions and two scalars. We choose the following tensor structures for the four-point function of two spin half fermions and two scalars $\langle\psi_1\psi_2O_3O_4\rangle$, as in [179],

$$\begin{aligned}
t^+_1 &= \frac{\langle S_1S_2\rangle}{\sqrt{X_{12}}}, \qquad & t^+_2 &= \frac{\langle S_1X_3X_4S_2\rangle}{\sqrt{X_{13}X_{34}X_{42}}}, \\
t^-_3 &= \frac{\langle S_1X_3S_2\rangle}{\sqrt{X_{13}X_{32}}}, \qquad & t^-_4 &= \frac{\langle S_1X_4S_2\rangle}{\sqrt{X_{14}X_{42}}}.
\end{aligned} \tag{4.22}$$

4.2.4 Four-point function of fermions

We present here the basis for the fermion four-point function that we shall use. We shall justify our choice with more details in sec. 4.5. In this case, the tensor structures are of the form $\langle S_i \cdots S_j\rangle \langle S_k \cdots S_l\rangle$. The idea of expressing the tensor structures (in embedding space notation) in a chosen conformal frame for relating (or showing the mutual independence of) different tensor structures is particularly useful in this context.

For four fermions there are sixteen independent tensor structures and we pick a basis with elements of definite parity. The parity even structures are taken to be,

$$\begin{aligned}
p_1^+ &= \frac{\langle S_1 S_2\rangle \langle S_3 S_4\rangle}{\sqrt{X_{12}X_{34}}}, & p_2^+ &= \frac{\langle S_1 S_2\rangle \langle S_3 X_1 X_2 S_4\rangle}{\sqrt{X_{12}^2 X_{13} X_{24}}}, \\
p_3^+ &= \frac{\langle S_1 X_3 \Gamma^A S_2\rangle \langle S_3 X_1 \Gamma_A S_4\rangle}{\sqrt{X_{13}X_{32}X_{31}X_{14}}}, & p_4^+ &= \frac{\langle S_1 \Gamma^A \Gamma^B S_2\rangle \langle S_3 \Gamma_A \Gamma_B S_4\rangle}{\sqrt{X_{12}X_{34}}}, \\
p_5^+ &= \frac{\langle S_1 X_3 S_2\rangle \langle S_3 X_1 S_4\rangle}{\sqrt{X_{13}^2 X_{14} X_{23}}}, & p_6^+ &= \frac{\langle S_1 X_3 S_2\rangle \langle S_3 X_2 S_4\rangle}{\sqrt{X_{23}^2 X_{13} X_{24}}}, \\
p_7^+ &= \frac{\langle S_1 X_4 S_2\rangle \langle S_3 X_1 S_4\rangle}{\sqrt{X_{13}X_{14}^2 X_{24}}}, & p_8^+ &= \frac{\langle S_1 X_4 S_2\rangle \langle S_3 X_2 S_4\rangle}{\sqrt{X_{14}X_{23}X_{24}^2}}.
\end{aligned} \tag{4.23}$$

The parity odd part of the basis can be taken to be composed of the following structures,

$$\begin{aligned}
p_9^- &= \frac{\langle S_1 S_2\rangle \langle S_3 X_1 S_4\rangle}{\sqrt{X_{12}X_{13}X_{14}}}, & p_{10}^- &= \frac{\langle S_1 S_2\rangle \langle S_3 X_2 S_4\rangle}{\sqrt{X_{12}X_{23}X_{24}}}, \\
p_{11}^- &= \frac{\langle S_1 X_3 S_2\rangle \langle S_3 S_4\rangle}{\sqrt{X_{13}X_{23}X_{34}}}, & p_{12}^- &= \frac{\langle S_1 X_4 S_2\rangle \langle S_3 S_4\rangle}{\sqrt{X_{14}X_{24}X_{34}}}, \\
p_{13}^- &= \frac{\langle S_1 \Gamma^A S_2\rangle \langle S_3 \Gamma_A X_1 S_4\rangle}{\sqrt{X_{12}X_{13}X_{14}}}, & p_{14}^- &= \frac{\langle S_1 \Gamma^A S_2\rangle \langle S_3 \Gamma_A X_2 S_4\rangle}{\sqrt{X_{12}X_{23}X_{24}}}, \\
p_{15}^- &= \frac{\langle S_1 \Gamma^A X_3 S_2\rangle \langle S_3 \Gamma_A S_4\rangle}{\sqrt{X_{13}X_{23}X_{34}}}, & p_{16}^- &= \frac{\langle S_1 \Gamma^A X_4 S_2\rangle \langle S_3 \Gamma_A S_4\rangle}{\sqrt{X_{14}X_{42}X_{34}}}.
\end{aligned} \tag{4.24}$$

Note that the basis presented above is different from the one that appears in the literature [164]. We provided further details in appendix. D including the change of basis that relates our basis (even part) in eq. (4.23) to the one in [164], and how crossing acts on our chosen basis in eq. (4.23) and eq. (4.24).

4.3 Definition of Mellin amplitude

After our discussion on tensor structures, we are equipped to define Mellin amplitudes for correlators of fermions and scalars. In general for a correlator of $2K$ fermions and M scalars ($2K + M = n$), we can define the Mellin amplitude (in the embedding space language) with the following set of Mellin-Barnes integrals,

$$\begin{aligned}
&\langle \Psi_1(X_1, S_1) \cdots \Psi_{2K}(X_{2K}, S_{2K})\, \Phi_{2K+1}(X_{2K+1}) \cdots \Phi_n(X_n)\rangle \\
= \quad &\sum_k \tilde{\mathbb{T}}_k \int [ds_{ij}] \prod_{1\le i<j}^{n} X_{ij}^{-s_{ij}-a_{ij;k}}\, \Gamma\left(s_{ij} + a_{ij;k} + n_{ij;k} + \frac{1}{2}\sum_{m=1}^{K} \delta_{i,2m-1}\delta_{j,2m}\right) \\
&\mathcal{M}_k(\{s_{ij}\}) \prod_{m=1}^{K} \frac{1}{\sqrt{X_{2m-1,2m}}},
\end{aligned} \tag{4.25}$$

The set $\{\mathcal{M}_k(s_{ij})\}$ is the Mellin amplitude. We demand the Mellin variables to satify the following constraints:

$$\tau_i - \sum_{l\neq i} s_{li} = 0 \quad \forall i. \tag{4.26}$$

In the equation above, the tensor structures $\tilde{\mathbb{T}}_i$ do not have a denominator (i.e. they are not normalized) unlike those in eq. (4.23) for example. The set $\{\tilde{\mathbb{T}}_i\}$ must form a basis of tensor structures for the given correlator and apart from being a Lorentz invariant in $d+2$ dimensions and satisfying the transversality condition, each $\tilde{\mathbb{T}}_i$ must satisfy the following homogeneity condition in S_i,

$$\tilde{\mathbb{T}}_k\left(b_1 S_1, \cdots, b_{2K} S_{2K}; X_1, \cdots, X_n\right) = \left(\prod_{i=1}^{2K} b_i\right) \tilde{\mathbb{T}}_k\left(S_1, \cdots, S_{2K}; X_1, \cdots, X_n\right). \tag{4.27}$$

$a_{ij;k}$ are numbers which determine the normalization of the tensor structure. Let us define,

$$\mathbb{T}_k = \tilde{\mathbb{T}}_k \prod_{1\leq i<j}^{n} X_{ij}^{-a_{ij;k}}. \tag{4.28}$$

Concretely, the numbers $a_{ij;k}$ are fixed by the requirement that given $\lambda_i = \sqrt{\sigma_i}$, the following must hold,

$$\mathbb{T}_k\left(\lambda_1 S_1, \cdots, \lambda_{2K} S_{2K}; \sigma_1 X_1, \cdots, \sigma_n X_n\right) = \mathbb{T}_k\left(S_1, \cdots, S_{2K}; X_1, \cdots, X_n\right). \tag{4.29}$$

τ_i is the twist of the operator at X_i. So $\tau_i = \Delta_i - \frac{1}{2}$ for $i \in \{1, \cdots, 2K\}$ and $\tau_j = \Delta_j$ for $j \in \{2K+1, \ldots, n\}$. Note that eq. (4.27) and eq. (4.29) together give a definition that is equivalent to eq. (4.11) for the (normalization of the) tensor structures $\mathbb{T}_k$. The tensor structures in eq. (4.22), eq. (4.23) and eq. (4.24) are normalized in this manner.

$n_{ij;k}$ are integers that we keep undetermined for now. The gamma functions in eq. (4.25) have been extracted in analogy with the case of scalars to simplify the asymptotics of the Mellin amplitude on the complex plane and the factorization formulae. In sec. 4.6 we shall be computing Mellin amplitudes in the large N limit of a strongly coupled CFT (through tree level Witten diagrams) and in sec. 4.7 we shall be computing Mellin amplitudes in a weakly interacting CFT. We shall choose $n_{ij;k}$ such that in either case the Mellin amplitude for the contact interaction are polynomials in the Mellin variables (constant for the contact Witten diagrams). This way, the Mellin amplitudes in the large N limit of the strongly coupled CFT (dual to a quantum field theory in AdS) encodes only the bulk dynamics. In the perturbative regime, the singularities of the Mellin amplitude do not carry information on the trivial composite operators.

It can be checked that the correlator in eq. (4.25) is consistent with the homogeneity condition eq. (4.10), given that eq. (4.27) and eq. (4.29) are satisfied. The conformality constraints imposed by eq. (4.26) in eq. (4.25) can be interpreted in terms of fictitious Mellin momenta k_i with $k_i \cdot k_j = s_{ij}$ and an on-shell condition $k_i^2 = -\tau_i$ as the overall conservation of Mellin momentum $\sum_i k_i = 0$. This is a generalization of the corresponding scenario for scalar correlator as discussed in sec. 4.1. This time, one can relate the Mellin variables to Mandelstam variables as

$$S_{i_1\cdots i_a} = -\left(k_{i_1} + \cdots + k_{i_a}\right)^2 = -2\sum_{l<k\leq a} s_{i_l i_k} + \sum_{j=1}^{a} \tau_{i_j}. \tag{4.30}$$

Since we have chosen to work in Minkowski spacetime, we shall always understand that $X_{ij} \to -(x_i - x_j)^2 + i\epsilon_{ij}$. The relative values of all the ϵ_{ij} is assumed to be consistent with the time ordering in the correlator.

In this paper, we shall mainly be focussing on the four-point function. We shall assume for simplicity that all operators of the same spin have different conformal dimensions. Let us describe concretely, the definition for the two kinds of four-point functions. Here we make a choice of $n_{ij;k}$. The Mellin amplitude for the four-point function of two fermions and two scalars is defined by the following,

$$\langle \Psi_1 \Psi_2 \Phi_3 \Phi_4 \rangle = \int [ds_{ij}] \prod_{i<j} (X_{ij})^{-s_{ij}} \frac{1}{\sqrt{X_{12}}} \left[\sum_{i=1}^{4} t_i \bar{M}_i \left(\{s_{ab}\}\right) \right]. \tag{4.31}$$

The Mellin variables satisfy the conformality constraints as mentioned in eq. (4.26).

The tensor structures t_i for this correlator are chosen in eq. (4.22). In eq. (4.31), the superscript from eq. (4.22) indicating the parity of the tensor structures t_i has been suppressed. Following Mack [79], we shall call $\bar{M} \equiv \{\bar{M}_i\}$ the reduced Mellin amplitude. In this case, we choose all the integers $n_{ij;k}$ to be zero. The relations between the Mellin amplitude $\{\mathcal{M}_i\}$ and the reduced Mellin amplitude $\{\bar{M}_i\}$ are given explicitly in appendix. E.1.

Similarly, we define the four-point function of fermions in the following way,

$$\langle \Psi_1 \Psi_2 \Psi_3 \Psi_4 \rangle = \int [ds_{ij}] \prod_{i<j} (X_{ij})^{-s_{ij}} \frac{1}{\sqrt{X_{12} X_{34}}} \left[\sum_{i=1}^{16} p_i \bar{M}_i \left(\{s_{ab}\}\right) \right]. \tag{4.32}$$

The tensor structures p_i are as in eq. (4.23), eq. (4.24). The choice of the integers $n_{ij;k}$ dictates the relation between the reduced Mellin amplitude $\{\bar{M}_i\}$ and the Mellin amplitude $\{\mathcal{M}_i\}$. We choose all the integers to be zero apart from the following,

$$n_{12;2} = n_{13;3} = n_{13;5} = n_{23;6} = n_{14;7} = n_{24;8} = -1\,. \tag{4.33}$$

We have spelled out the relations explicitly in appendix. E.2.

4.4 Pole structure: fermion-scalar four-point function

In this section, we will look at the pole structure of the mixed fermion scalar four-point function in the direct and the crossed channels.

4.4.1 Direct channel

The mixed fermion scalar four-point function can be expressed in the following manner.

$$\begin{aligned}
\langle\psi_1\psi_2\phi_3\phi_4\rangle &= \left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}\frac{\mathcal{A}(u,v)}{(X_{12})^{\frac{\Delta_1+\Delta_2}{2}}(X_{34})^{\frac{\Delta_3+\Delta_4}{2}}}, \qquad (4.34)\\
\mathcal{A}(u,v) &= \int_{c_s-i\infty}^{c_s+i\infty}\frac{ds}{4\pi i}\int_{c_t-i\infty}^{c_t+i\infty}\frac{dt}{4\pi i}\left[\sum_{i=1}^{4}t_i\bar{M}_i(s,t)\right]u^{\frac{s}{2}}v^{-\frac{s+t-\tau_1-\tau_4}{2}}, \qquad (4.35)\\
s=-(k_1+k_2)^2 &= \tau_1+\tau_2-2s_{12}, \qquad t=-(k_1+k_3)^2=\tau_1+\tau_3-2s_{13}.
\end{aligned}$$

We wish to compare eq. (4.35) with the contribution to $\mathcal{A}(u,v)$ from a single operator exchanged in the direct channel. For this, one can do a "dimensional analysis" to check the power law behavior of $\mathcal{A}(u,v)$ in u in the OPE limit $(u,v)\to(0,1)$ (with $\frac{1-v}{\sqrt{u}}$ held fixed). We have explicitly checked the leading behavior of the conformal blocks using the differential operators presented in [164] that enable one to obtain these direct channel blocks from the corresponding blocks for scalar four-point function. The contribution from one operator exchanged via the OPE has also been presented in general in [187] for external operators with any value of spin. In this paper the Gelfand-Tsetlin basis for $Spin(d)$ representations has been used. Our basis is defined by our choice of gamma matrices (as in [164]) and we are using three-point structures of definite parity unlike in [187]. The operators contributing to the direct channel block expansion are those that appear in both the OPE of two scalars and that of two spin-half fermions, and hence are integer spin operators in symmetric traceless representations of the Lorentz group.

Let $\mathcal{A}(u,v)=\sum_i t_i\mathcal{A}_i(u,v)$. The three-point function of two fermions and an integer spin operator has in general four independent tensor structures eq. (4.19) and hence four structure constants. Consequently each $\mathcal{A}_i(u,v)$ will in general receive contributions from four different conformal partial waves $g^a_{\Delta,l}$ (with covariant pre-factors stripped off). Let $g^{i,a}_{\Delta,l}$ be the contribution of $g^a_{\Delta,l}$ to $\mathcal{A}_i$. Here "a" lables the four tensor structures in the three-point function $\langle\psi_1\psi_2\mathcal{O}_l\rangle$.

Let us recall from eq. (4.22) that t_1,t_2 are parity even and t_3,t_4 are parity odd. Also from eq. (4.19), r_1,r_2 are parity even and r_3,r_4 are parity odd. Considering this and the explicit form of the three-point structures, we see that the only non-zero $g^{i,a}_{\Delta,l}$ are $g^{1,1}_{\Delta,l}$, $g^{1,2}_{\Delta,l}$, $g^{2,2}_{\Delta,l}$, $g^{3,3}_{\Delta,l}$, $g^{3,4}_{\Delta,l}$, $g^{4,3}_{\Delta,l}$ and $g^{4,4}_{\Delta,l}$. For $l=0$, the only non-zero ones are $g^{1,1}_{\Delta,0}\equiv g^{1,+}_{\Delta,0}$, $g^{3,3}_{\Delta,0}\equiv g^{3,-}_{\Delta,0}$ and $g^{4,3}_{\Delta,0}\equiv g^{4,-}_{\Delta,0}$.

We summarise the limiting behavior of $g^{i,a}_{\Delta,l}$ in the OPE limit here. This is generically given by some combination of Gegenbauer polynomials. For $l\geq 1$,

$$\begin{aligned}
\mathcal{A}_1 &\supset \lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{1,1}_{s,\Delta,l}+\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{1,2}_{s,\Delta,l} \qquad (4.36)\\
&\approx u^{\frac{\Delta}{2}}\sum_{k=0}^{\left\lfloor\frac{l-2}{2}\right\rfloor}\left(\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{1,k}_1+\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}^{2,k}_1\right)\left(\frac{v-1}{2\sqrt{u}}\right)^{l-2k}+\cdots,
\end{aligned}$$

$$\mathcal{A}_2 \supset \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{2,2}_{s,\Delta,l} \approx \lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}u^{\frac{\Delta}{2}}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}\mathcal{K}_2^k\left(\frac{v-1}{2\sqrt{u}}\right)^{l-1-2k} + \cdots\,, \tag{4.37}$$

$$\begin{aligned}\mathcal{A}_3 \supset\;& \lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{3,3}_{s,\Delta,l} + \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{3,4}_{s,\Delta,l}\\ \approx\;& u^{\frac{\Delta}{2}}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}_3^{3,k}\left(\frac{v-1}{2\sqrt{u}}\right)^{l-2k}\\ &+ u^{\frac{\Delta-1}{2}}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}_3^{4,k}\left(\frac{v-1}{2\sqrt{u}}\right)^{l-1-2k} + \cdots\,,\end{aligned} \tag{4.38}$$

$$\begin{aligned}\mathcal{A}_4 \supset\;& \lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{4,3}_{s,\Delta,l} + \lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}g^{4,4}_{s,\Delta,l}\\ \approx\;& u^{\frac{\Delta}{2}}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}_4^{3,k}\left(\frac{v-1}{2\sqrt{u}}\right)^{l-2k}\\ &+ u^{\frac{\Delta-1}{2}}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{K}_4^{4,k}\left(\frac{v-1}{2\sqrt{u}}\right)^{l-1-2k} + \cdots\,.\end{aligned} \tag{4.39}$$

$\lambda^a_{\psi_1\psi_2\mathcal{O}_l}$ are the structure constants of the three-point function $\langle\psi_1\psi_2\mathcal{O}_l\rangle$ associated to the tensor structure $r_{d,i}$ as in eq. (4.19). $\mathcal{K}_i^{j,k}$ are constants.

For a scalar exchange, matters simplify as $\lambda^1_{\psi_1\psi_2\mathcal{O}_0} \equiv \lambda^+_{\psi_1\psi_2\mathcal{O}}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_0} \equiv \lambda^-_{\psi_1\psi_2\mathcal{O}}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_0} \equiv 0$ and $\lambda^4_{\psi_1\psi_2\mathcal{O}_0} \equiv 0$. So we have,

$$\mathcal{A}_1 \supset \lambda^+_{\psi_1\psi_2\mathcal{O}}\lambda_{\mathcal{O}\phi_3\phi_4}\mathcal{K}_1^{1,0}u^{\frac{\Delta}{2}} + \cdots\,, \tag{4.40}$$

$$\mathcal{A}_3 \supset \lambda^-_{\psi_1\psi_2\mathcal{O}}\lambda_{\mathcal{O}\phi_3\phi_4}\mathcal{K}_3^{3,0}u^{\frac{\Delta}{2}} + \cdots\,, \tag{4.41}$$

$$\mathcal{A}_4 \supset \lambda^-_{\psi_1\psi_2\mathcal{O}}\lambda_{\mathcal{O}\phi_3\phi_4}\mathcal{K}_4^{3,0}u^{\frac{\Delta}{2}} + \cdots\,. \tag{4.42}$$

Comparing eq. (4.36) - eq. (4.39) and eq. (4.40) - eq. (4.42) with eq. (4.35), we can deduce the pole structure for $\bar{M}_i$. The Mellin amplitude $\{\mathcal{M}_i\}$ can be obtained from the reduced Mellin amplitude $\{\bar{M}_i\}$ as shown in eq. (E.1) and thus we obtain the pole structure of the Mellin amplitude in this channel as summarised in table. 4.1. When the exchanged operator is a scalar $l = 0$, we should take all structure constants apart from $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}$, $\lambda_{\mathcal{O}_l\phi_3\phi_4}$ to be zero.

$k = 0$ corresponds to the exchange of the primary and the leading twist descendants while $k > 0$ corresponds to the descendants with higher values of twist. Generically the singular terms in each component of the Mellin amplitude are of the following form,

$$\frac{\lambda^a_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{F}_{l,k}(t)}{s-\tau-2k}. \tag{4.43}$$

Component of M.A.	Location of Poles	Residues $\sim$
$\mathcal{M}_1$	$\tau+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$
$\mathcal{M}_2$	$\tau+1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$
$\mathcal{M}_3$	$\tau+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$
$\mathcal{M}_4$	$\tau+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}$

Table 4.1: Fermion-scalar four-point function: Direct channel poles

$\mathcal{F}_{l,k}(t)$ can be expected to be a polynomial in t whose degree is determined by spin l of the exchanged operator. The exact form of these polynomials associated to Mellin amplitudes involving fermions have not yet been derived.

The Mellin amplitude of the three-point function $\langle\psi_1\psi_2\mathcal{O}_l\rangle$ has four components, and each one is a constant proportional to the corresponding structure constant $\lambda^a_{\psi_1\psi_2\mathcal{O}_l}$. The Mellin amplitude associated with $\langle\mathcal{O}_l\phi_3\phi_4\rangle$ is just a constant proportional to $\lambda_{\mathcal{O}_l\phi_3\phi_4}$. Therefore from eq. (4.43), it is clear that each component of the Mellin amplitude associated with the four-point function $\langle\psi_1\psi_2\phi_3\phi_4\rangle$ factorizes on the poles listed above onto products of components of Mellin amplitudes of the corresponding three-point functions.

4.4.2 Crossed channel

Now we consider the exchange of operators in the crossed channel, in particular the OPE channel $13-24$. The four-point function can be expressed as follows,

$$\langle\psi_1\psi_2\phi_3\phi_4\rangle_c = \left(\frac{X_{34}}{X_{14}}\right)^{\frac{\Delta_{13}}{2}}\left(\frac{X_{14}}{X_{12}}\right)^{\frac{\Delta_{24}}{2}}\frac{\tilde{v}^{\frac{\Delta_{13}}{2}}}{X_{13}^{\frac{\Delta_1+\Delta_3}{2}}X_{24}^{\frac{\Delta_2+\Delta_4}{2}}}\sum_{i=1}^{4}t_i\tilde{A}_i\left(\tilde{u},\tilde{v}\right), \tag{4.44}$$

$$\tilde{u}=\frac{X_{13}X_{24}}{X_{12}X_{34}}, \qquad \tilde{v}=\frac{X_{14}X_{23}}{X_{12}X_{34}}, \qquad \Delta_{ij}=\Delta_i-\Delta_j,$$

$$\tilde{A}_i\left(\tilde{u},\tilde{v}\right) = \int\frac{dt}{4\pi i}\int\frac{ds}{4\pi i}\bar{M}_i\left(s,t\right)\tilde{u}^{\frac{t+\frac{1}{2}}{2}}\tilde{v}^{\frac{s+t+\frac{1}{2}-\Delta_1-\Delta_4}{2}}. \tag{4.45}$$

The operators contributing to the block expansion in the crossed channel are fermionic operators. Once again, we shall compare eq. (4.45) with the leading behavior of the corresponding blocks in the OPE limit $x_1\to x_3$. These blocks are also a type of “seed-blocks” in three dimensions, and have been computed in [179, 188]. We have chosen the same tensor structures as they have for the relevant three-point functions eq. (4.21) and also the same tensor structures for the four-point function.

Three-point functions of one spin-half fermion, one scalar and one generic fermion have one parity odd and one parity even tensor structure eq. (4.21). Hence each $\tilde{\mathcal{A}}_i$ will receive contributions from four different blocks $g^{i,\pm\pm}$. Let $g^{i,jk}_{\Delta,l}$ be the contribution to $\tilde{\mathcal{A}}_i$ from the block associated with the fusion of tensor structures r^j_{cr} and r^k_{cr} eq. (4.21) of the three-point functions. Therefore we can see from parity selection rules that the only non-zero $g^{i,jk}_{\Delta,l}$ are $g^{1,++}_{\Delta,l}$, $g^{1,--}_{\Delta,l}$, $g^{2,++}_{\Delta,l}$, $g^{2,--}_{\Delta,l}$, $g^{3,+-}_{\Delta,l}$, $g^{3,-+}_{\Delta,l}$, $g^{4,+-}_{\Delta,l}$ and $g^{4,-+}_{\Delta,l}$.

The mixed fermion scalar conformal blocks can be found in [179]. These blocks are expressed in invariants r, η introduced in [16]:

$$\tilde{u} = \frac{16r^2}{(1+r^2-2r\eta)^2}, \qquad \tilde{v} = \frac{(1+r^2+2r\eta)^2}{(1+r^2-2r\eta)^2}. \tag{4.46}$$

The OPE limit in these coordinates is now given by $r \to 0$ with η held constant. One can check that for small r, $\tilde{u} \approx r^2$ and $\eta \approx -\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}$. The leading behavior of these blocks as $r \to 0$ is given by,

$$\begin{aligned}
g^{(1,++)}_{\Delta,l}(r,\eta) &= -r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(1,--)}_{\Delta,l}(r,\eta) &= -r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(2,++)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(2,--)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(3,+-)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(3,-+)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(4,+-)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) + P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right), \\
g^{(4,-+)}_{\Delta,l}(r,\eta) &= r^{\Delta}\left(P^{(0,1)}_{l-\frac{1}{2}}(\eta) - P^{(1,0)}_{l-\frac{1}{2}}(\eta)\right) + O\left(r^{\Delta+1}\right).
\end{aligned}$$

$P^{(\alpha,\beta)}_n(z)$ are Jacobi polynomials. We note the symmetry property of Jacobi polynomials,

$$P^{(\alpha,\beta)}_n(-z) = (-1)^n P^{(\beta,\alpha)}_n(z). \tag{4.47}$$

eq. (4.47) implies that $P^{(\alpha,\beta)}_n(z) + P^{(\beta,\alpha)}_n(z)$ has only even powers of z for even n and only odd powers of z for odd n; and $P^{(\alpha,\beta)}_n(z) - P^{(\beta,\alpha)}_n(z)$ has only odd powers of z for even n and even powers of z for odd n. Considering this, and the series expansion of the Jacobi polynomials, we can express the leading behavior of the blocks (for $l > \frac{1}{2}$ and with $l = l - \frac{1}{2}$) in the following manner ,

$$
\begin{aligned}
g^{(1,++)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx -\tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lfloor \frac{l}{2} \right\rfloor} H^{+(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k} + \cdots , \qquad (4.48)\\
g^{(1,--)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx -\tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lceil \frac{l}{2} \right\rceil -1} H^{-(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k-1} + \cdots ,\\
g^{(2,++)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lceil \frac{l}{2} \right\rceil -1} H^{-(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k-1} + \cdots ,\\
g^{(2,--)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lfloor \frac{l}{2} \right\rfloor} H^{+(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k} + \cdots ,\\
g^{(3,+-)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lceil \frac{l}{2} \right\rceil -1} H^{-(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k-1} + \cdots ,\\
g^{(3,-+)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lfloor \frac{l}{2} \right\rfloor} H^{+(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k} + \cdots ,\\
g^{(4,+-)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lfloor \frac{l}{2} \right\rfloor} H^{+(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k} + \cdots ,\\
g^{(4,-+)}_{\Delta,l}(\tilde{u},\tilde{v}) &\approx \tilde{u}^{\frac{\Delta}{2}} \sum_{k=0}^{\left\lceil \frac{l}{2} \right\rceil -1} H^{-(0,1)}_{l,k} \left(\frac{1-\tilde{v}}{2\sqrt{\tilde{u}}}\right)^{l-2k-1} + \cdots .
\end{aligned}
$$

$H^{\pm(\alpha,\beta)}_{n,k}$ are coefficients in the series expansions of $P^{(\alpha,\beta)}_n(z) \pm P^{(\beta,\alpha)}_n(z)$. For $l = \frac{1}{2}$, we have,

$$
\begin{aligned}
g^{(1,++)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx -2\tilde{u}^{\frac{\Delta}{2}} + \cdots , & g^{(1,--)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx -2\tilde{u}^{\frac{\Delta+1}{2}} + \cdots , \qquad (4.49)\\
g^{(2,++)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx 2\tilde{u}^{\frac{\Delta+1}{2}} + \cdots , & g^{(2,--)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx 2\tilde{u}^{\frac{\Delta}{2}} + \cdots ,\\
g^{(3,+-)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx 2\tilde{u}^{\frac{\Delta+1}{2}} + \cdots , & g^{(3,-+)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx 2\tilde{u}^{\frac{\Delta}{2}} + \cdots ,\\
g^{(4,+-)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx 2\tilde{u}^{\frac{\Delta}{2}} + \cdots , & g^{(4,-+)}_{\Delta,\frac{1}{2}}(\tilde{u},\tilde{v}) &\approx 2\tilde{u}^{\frac{\Delta+1}{2}} + \cdots .
\end{aligned}
$$

Component of M.A.	**Location of Poles**	**Residues** $\sim$
$\mathcal{M}_1$	$t = \tau + 2k$	$\lambda^{+}_{\psi_1\phi_3\Psi_l}\lambda^{+}_{\Psi_l\phi_4\psi_2}$
	$t = \tau + 1 + 2k$	$\lambda^{-}_{\psi_1\phi_3\Psi_l}\lambda^{-}_{\Psi_l\phi_4\psi_2}$
$\mathcal{M}_2$	$t = \tau + 1 + 2k$	$\lambda^{+}_{\psi_1\phi_3\Psi_l}\lambda^{+}_{\Psi_l\phi_4\psi_2}$
	$t = \tau + 2k$	$\lambda^{-}_{\psi_1\phi_3\Psi_l}\lambda^{-}_{\Psi_l\phi_4\psi_2}$
$\mathcal{M}_3$	$t = \tau + 1 + 2k$	$\lambda^{+}_{\psi_1\phi_3\Psi_l}\lambda^{-}_{\Psi_l\phi_4\psi_2}$
	$t = \tau + 2k$	$\lambda^{-}_{\psi_1\phi_3\Psi_l}\lambda^{+}_{\Psi_l\phi_4\psi_2}$
$\mathcal{M}_4$	$t = \tau + 2k$	$\lambda^{+}_{\psi_1\phi_3\Psi_l}\lambda^{-}_{\Psi_l\phi_4\psi_2}$
	$t = \tau + 1 + 2k$	$\lambda^{-}_{\psi_1\phi_3\Psi_l}\lambda^{+}_{\Psi_l\phi_4\psi_2}$

Table 4.2: Fermion-scalar four-point function: Crossed channel poles.

Let $\lambda^{\pm}_{\psi\phi\Psi_l}$ be the structure constant associated with the term with tensor structure $r^{\pm}_{cr}$ (see eq. (4.21)) in the three-point function $\langle\psi\phi\Psi_l\rangle$. Comparing eq. (4.48) and eq. (4.49) with eq. (4.44) and eq. (4.45), we can conclude that the reduced Mellin amplitude and consequently the Mellin amplitude has the poles in t as summarised in table. 4.2 for the exchange of fermionic operator Ψ_l with twist τ.

We see a novelty in the pole structure here. Each component of the Mellin amplitude has two series of poles for each primary exchanged. It is clear that each component of the Mellin amplitude $\bar{M}_i$ factorizes at the poles onto components of the Mellin amplitudes of the corresponding three-point functions as described above.

There are also poles in the Mellin amplitude in the u-channel. The u-channel is related to the s- and t-channel by the relation $u = \sum_i \tau_i - s - t$. These correspond to operators exchanged in the OPE channel 14-23. The location of these poles can be worked out from the preceeding discussion. We state the results directly in table. F.1 in appendix. F.1.

4.5 Pole structure: fermion four-point function

The Mellin amplitude eq. (4.6) associated with the correlator $\langle\phi_1\phi_2\phi_3\phi_4\rangle$ has s-channel poles at $s = \tau + 2k$ for each operator with twist τ contributing to the s-channel conformal block expansion of the correlator. The residue at the pole is $\lambda_{\phi_1\phi_2\mathcal{O}_l}\lambda_{\mathcal{O}_l\phi_3\phi_4}\mathcal{Q}_{l,k}(t)$, l being the spin of the exchanged operator. $\mathcal{Q}_{l,k}$ are polynomials in t of degree l. One way to explain this analyticity property is in terms of the expansion of the conformal block $G_{\Delta,l}$ around the

OPE limit [173],

$$G_{\Delta,l} = u^{\frac{\Delta-l}{2}} \sum_{k=0}^{\infty} u^k g_k(v), \tag{4.50}$$

where $g_k(v)$ has a power series expansion in $1-v$.

For the correlator $\langle \psi_1\psi_2\psi_3\psi_4\rangle$, the nature of the conformal blocks depends on the basis of tensor structures. As mentioned earlier in sec. 4.2, a generic basis of tensor structures may lead to the conformal blocks having spurious singularities. We will choose a basis such that each conformal block can be expanded around the OPE limit as follows,

$$\sum_{i=1}^{I} u^{\frac{\tau-a_i}{2}} \sum_{k=0}^{\infty} u^k \tilde{g}_k(v). \tag{4.51}$$

Here I is some finite integer greater than zero, $a_i < \tau$ are integers and $\tilde{g}_k$ has a power series expansion in $1-v$. These would ensure that each component of the Mellin amplitude has finitely many series of poles corresponding to each exchanged primary and the residue at each pole is a product of the relevant structure constants and a polynomial whose degree is determined by the spin l. eq. (4.23), eq. (4.24) is one possible choice of such a basis $\{p_i\}$.

We state the results for the pole structure here. Corresponding to each integer spin l primary $\mathcal{O}_l$ of twist τ contributing to the direct channel conformal block expansion of the correlator, the Mellin amplitude has poles and residues as summarised in table. 4.3.

When the exchanged operator is a scalar $l=0$, we should take all structure constants apart from $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}$, $\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$ and $\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$ to be zero. The poles in the crossed channels can also be worked out. We state the results in appendix. F.2.

4.6 Mellin amplitudes for Witten diagrams

The AdS/CFT correspondence is a conjectured duality between String Theories in $d+1$ dimensional AdS spacetime and CFTs living on its d dimensional boundary. When the bulk spacetime is weakly curved and the bulk theory is well approximated by the supergravity limit, we can use Witten diagrams to compute correlation functions in the dual strongly interacting CFT. These computations are quite cumbersome in position space. In the Mellin representation, they are simplified greatly [81, 86, 87, 212] and the corresponding Mellin amplitudes can be concretely related to scattering amplitudes in QFT in $d+1$ dimensions through the so-called "flat-space limit" [81, 84, 85].

In this section, we shall present a few results for tree-level Witten diagrams with fermionic legs which serve to illustrate some of the general features discussed in the sections 4.4 and 4.5. The calculations are simply reduced to calculations of scalar Witten diagrams [213, 214], the results for which are available [81, 86]. Hence we do not need to set up these calculations in embedding space notation. We shall however present results in embedding space notation in order to relate them to the tensor structures in sec. 4.2. We begin the discussion with a short review of fermions in the AdS/CFT correspondence and then move on to computing the Witten diagrams.

Component of M.A.	**Location of Poles**	**Residues $\sim$**
$\mathcal{M}_1$	$s=\tau+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_2$	$s=\tau+1+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_7$, $\mathcal{M}_8$	$s=\tau-1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_4$	$s=\tau+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_9$, $\mathcal{M}_{10}$	$s=\tau+2k$	$\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^1_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_{11}$, $\mathcal{M}_{12}$	$s=\tau+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_4}$ $\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_{13}$, $\mathcal{M}_{14}$	$s=\tau+1+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+2+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
$\mathcal{M}_{15}$, $\mathcal{M}_{16}$	$s=\tau+1+2k$	$\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^2_{\psi_1\psi_2\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_4}$
	$s=\tau+2+2k$	$\lambda^3_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$, $\lambda^4_{\psi_1\psi_2\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_4}$

Table 4.3: Fermion four-point function: Direct channel poles.

In the diagrams in this section, solid lines with arrows denote fermion propagators and solid lines without arrows denote scalar propagators. Although the results presented in this section are not strictly dimension dependent, when we refer to our discussion on the pole structure of the Mellin amplitude and the parity of associated tensor structures, we shall be implicitly assuming that we are in dimension three.

4.6.1 Brief review of Fermions in AdS

We wish to compute correlators in a strongly interacting CFT that admits an expansion in $\frac{1}{N}$ expansion in d dimensions from the dynamics of the dual theory in the $d+1$ dimensional bulk. The basic principle behind this is as follows: Let the field ϕ be the bulk dual to a given operator $\mathcal{O}$ of the boundary CFT. The (appropriately defined) boundary restriction ϕ_0 of ϕ acts as the source for $\mathcal{O}$. The partition function of the bulk theory can be computed in terms of the boundary field ϕ_0 using the Green's functions in the bulk. The AdS/CFT correspondence states that the partition function of the bulk and the boundary theories are equal and thus one can calculate correlation functions from this bulk partition function by taking derivatives with respect to ϕ_0.

In the planar limit of the strongly interacting boundary CFT, the bulk partition function is dominated by the classical value as given by $\phi_{\rm cl}$ that obeys the equation of motion (e.o.m.). Thus, the CFT correlation functions are given by tree level Witten diagrams in the bulk. This is the regime we shall focus on in this section. To evaluate the bulk action $S[\phi_{\rm cl}]$ in practice, we can expand $\phi_{\rm cl}$ in a perturbative expansion around the boundary field [214]. From this, we can obtain planar CFT correlators as follows:

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_n)\rangle \approx e^{S[\phi_{\rm cl}]}\frac{\delta}{\delta\phi_0(x_1)}\cdots\frac{\delta}{\delta\phi_0(x_n)}e^{-S[\phi_{\rm cl}]}\bigg|_{\phi_0=0}. \tag{4.52}$$

To approximate the bulk partition function by the corresponding classical value, it is crucial to ensure the stationarity of the action at the classical path. Therefore, as is common for spaces with boundaries, an analysis of boundary terms to supplement the Dirac action S_D is necessary. The surface term $S_{\rm F}$ that is added to the action should respect the symmetries of AdS and ensure the stationarity of the action on the classical solution.

For example, Yukawa theory in AdS described by the action [213, 215–217]:

$$\begin{aligned} S[\psi,\bar\psi,\phi] &= S_{\rm D}+S_{\rm KG}+S_{\rm int}+S_{\rm F}\,, \\ &= \int_M d^{d+1}z\,\sqrt{g}\left[\bar\psi\left(\not{D}-m\right)\psi+\frac{1}{2}\left((\nabla_\mu\phi)^2+M^2\phi^2\right)+\lambda\phi\bar\psi\psi\right] \\ &\quad+\int_{\partial M_\epsilon} d^d\vec{x}\,\sqrt{h_\epsilon}\bar\psi\psi, \end{aligned} \tag{4.53}$$

S_{KG} include the kinetic and mass terms of the scalar field ϕ and $S_{\rm int}$ is the interaction term. Here, we are on the Poincaré patch of AdS, with the line element:

$$ds^2=\frac{1}{z_0^2}\left(dz_0^2+d\vec{z}^2\right)=\frac{1}{z_0^2}dz^\mu dz_\mu.$$

$h_{\epsilon;ij}$ in eq. (4.53) is the induced metric on the surface ∂M_ϵ. ∂M_ϵ is the regularized boundary of the AdS space M, which approches the boundary for $z_0=\epsilon\to 0$ [213, 215–217].

Let us briefly review the behavior of spinors near the boundary of AdS following [216]. We can assume, without loss of any generality, that the mass $m \geq 0$ of the spinor is non-negative. Let us also drop the subscript "cl" for classical fields now. The classical solutions $\psi(z) = \psi^+(z) + \psi^-(z)$ to the Dirac equation in AdS have the following behavior close to the boundary:

$$\begin{aligned} \psi^-(z) &= z_0^{\frac{d}{2}-m} \psi_0^-(\vec{z}) + O(z_0^{\frac{d}{2}-m+1}) \,, \\ \psi^+(z) &= z_0^{\frac{d}{2}+m} \psi_0^+(\vec{z}) + O(z_0^{\frac{d}{2}+m+1}) \,. \end{aligned} \tag{4.54}$$

where $\psi^+(z)$ and $\psi^-(z)$ are eigenfunctions of $\Gamma^{(c)}$ in the bulk: $\Gamma^{(c)}\psi^\pm(z) = \pm\psi^\pm(z)$. This shows that for positive mass (as assumed), ψ_0^- is the leading contribution if one approaches the boundary[2].

Furthermore, we also have to demand regularity of the solutions in the bulk upto $z_0 \to \infty$. This gives further relations between ψ_0^- and ψ_0^+ and similarly between $\bar{\psi}_0^+$ and $\bar{\psi}_0^-$. All in all, this establishes that the boundary data is specified entirely by ψ_0^- (and similarly $\bar{\psi}_0^+$) when $m \geq 0$. For negative mass, the analysis and the leading behavior at the boundary is switched to the spinor with the opposite $\Gamma^{(c)}$ eigenvalue. Thus, when the boundary is odd dimensional, the boundary restriction of a bulk Dirac spinor is a Dirac spinor of the boundary CFT, and when the boundary is even dimensional, the boundary value of a bulk Dirac spinor is a Weyl spinor of the boundary theory.

The bulk field can be solved in orders of the coupling constant λ through recursion relations of the form:

$$\begin{aligned} \phi(z) &= \phi_\epsilon^{(0)}(z) - \lambda \int d^{d+1}w \sqrt{g} G_\epsilon(z,w) \bar{\psi}(w)\psi(w), \\ \psi(z) &= \psi_\epsilon^{(0)}(z) - \lambda \int d^{d+1}w \sqrt{g} S_\epsilon(z,w)\phi(w)\psi(w), \\ \bar{\psi}(z) &= \bar{\psi}_\epsilon^{(0)}(z) - \lambda \int d^{d+1}w \sqrt{g} \bar{\psi}(w)\phi(w) S_\epsilon(z,w). \end{aligned} \tag{4.55}$$

Here $\phi_\epsilon^{(0)}$, $\psi_\epsilon^{(0)}$ and $\bar{\psi}_\epsilon^{(0)}$ denote the regularized solutions to the e.o.m in free theory. $G_\epsilon(z,w)$ and $S_\epsilon(z,w)$ are the regularized scalar and spinorial bulk-to-bulk operators [136, 215]. As mentioned before, the regularized free theory solutions can in turn be obtained from the boundary restrictions in the limit $\epsilon \to 0$ as follows:

$$\begin{aligned} \phi^{(0)} &= \lim_{\epsilon\to 0} \phi_\epsilon^{(0)}(z) = \int d^d\vec{x}\; K_{\Delta_s}(z,\vec{x})\phi_0(\vec{x}), \qquad (4.56) \\ \psi^{(0)} &= \lim_{\epsilon\to 0} \psi_\epsilon^{0}(z) = -\int d^d\vec{x}\; \Sigma_\Delta(z,\vec{x})\, \psi_0^-(\vec{x}) \quad \text{with } \Sigma_\Delta(z,\vec{x}) = \frac{\Gamma_\mu (z^\mu - x^\mu)}{\sqrt{z_0}} K_{\Delta+\frac{1}{2}}(z,\vec{x})\, \mathcal{P}^-, \\ \bar{\psi}^{(0)} &= \lim_{\epsilon\to 0} \bar{\psi}_\epsilon^{0}(z) = \int d^d\vec{x}\; \bar{\psi}_0^+(\vec{x})\, \bar{\Sigma}_\Delta(z,\vec{x}) \quad \text{with } \bar{\Sigma}_\Delta(z,\vec{x}) = \mathcal{P}^+ \frac{\Gamma_\mu (z^\mu - x^\mu)}{\sqrt{z_0}} K_{\Delta+\frac{1}{2}}(z,\vec{x}). \end{aligned}$$

Here $K_\Delta(z,\vec{x})$ and $\Sigma_\Delta(z,\vec{x})$ are the scalar and fermionic bulk-to-boundary propagator, respectively (see [213]). These have been labelled with the dimension of the dual boundary

[2] $\Gamma^{(c)}$ is the chirality operator (usually denoted as Γ^5 in four spacetime dimensions). Note that in odd dimensions, $\Gamma^{(c)}$ is part of the Clifford algebra as $\Gamma^{d-1} = \Gamma^{(c)}$ and hence its eigenvalue can longer be used as a quantum number. This however does not affect the present discussion.

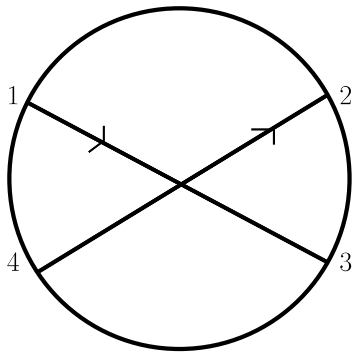


Figure 4.1: Mixed fermion-scalar four-point contact Witten diagram.

CFT operator. Γ^μ are gamma matrices of the bulk and so are the chirality projectors $\mathcal{P}^\pm = (1 \pm \Gamma^{(c)})/2$. The dimension of the boundary scalar operator dual to ϕ satisfies $\Delta_s (\Delta_s - D) = M^2$ [10] and the analogous relation for the spinor fields is $\Delta = m + \frac{d}{2}$ [215, 217].

Using eq. (4.55) and eq. (4.56) with $\epsilon \to 0$, the action can be written in a perturbation series in the coupling constant λ in terms of the boundary fields ϕ_0, ψ_0^- and $\bar{\psi}_0^+$. Thereafter, eq. (4.52) simply gives us the planar correlators in the boundary CFT.

4.6.2 Contact Witten diagrams

Let us consider the mixed fermion-scalar four-point contact Witten diagram as shown in fig. 4.1. As described in sec. 4.6.1, bulk-to-boundary spinor propagators can be obtained from bulk-to-boundary scalar propagators as follows,

$$\begin{aligned}\Sigma_\Delta (z, \vec{x}) &= \frac{\Gamma_\mu (z^\mu - x^\mu)}{\sqrt{z_0}} K_{\Delta+\frac{1}{2}} (z, \vec{x}) \mathcal{P}^-, \\ \bar{\Sigma}_\Delta (z, \vec{x}) &= \mathcal{P}^+ \frac{\Gamma_\mu (z^\mu - x^\mu)}{\sqrt{z_0}} K_{\Delta+\frac{1}{2}} (z, \vec{x}) .\end{aligned} \tag{4.57}$$

$\mathcal{P}^\pm$, Γ^μ are projectors and gamma matrices of the bulk spacetime. Using eq. (4.57), we can express the product of the two fermionic bulk-to-boundary propagators $\bar{\Sigma}_{\Delta_1}$ and Σ_{Δ_2} as a product of two scalar bulk-to-boundary propagators $K_{\Delta_1+\frac{1}{2}}$ and $K_{\Delta_2+\frac{1}{2}}$ while extracting out a tensor structure:

$$\bar{\Sigma}_{\Delta_1}(z, \vec{x}_1)\Sigma_{\Delta_2}(z, \vec{x}_2) = \left(\vec{x}_{12}^\mu \Gamma_\mu \mathcal{P}^-\right) K_{\Delta_1+\frac{1}{2}}(z, \vec{x}_1) K_{\Delta_2+\frac{1}{2}}(z, \vec{x}_2). \tag{4.58}$$

When contracted with polarization spinors localized on the boundary, $\vec{x}_{12}^\mu \Gamma_\mu \mathcal{P}^-$ is equivalent to $\vec{x}_{12}^a \gamma_a \equiv \not{x}_{12}$ (contracted with polarization spinors of the boundary) where γ_a are gamma matrices of the boundary spacetime. Thus the fermion-scalar four-point contact Witten diagram can be simply obtained from a scalar Witten diagram as follows:

$$B^{\bar{\psi}_1 \psi_2}_{\phi_3 \phi_4} = \langle S_1 S_2 \rangle \int_{AdS} dZ \prod_{i=1}^{2} K_{\Delta_i + \frac{1}{2}}(Z, X_i) \prod_{i=3}^{4} K_{\Delta_i}(Z, X_i). \tag{4.59}$$

We return to using embedding space language in eq. (4.59) in order to see the tensor structures in the form as discussed in sec. 4.2. Z is a bulk point and X_i are points on the boundary. The integral in eq. (4.59) is the expression of a contact Witten diagram of four scalars. The scalar bulk-to-boundary propagator is given by,

$$K_\Delta(Z,X) = \frac{C_\Delta}{(-2Z\cdot X)^\Delta}\,, \qquad C_\Delta = \frac{\Gamma(\Delta)}{2\pi^h\Gamma\left(\Delta+1-\frac{d}{2}\right)}\,. \tag{4.60}$$

Using eq. (4.60), the correlator in eq. (4.59) can be evaluated and expressed in Mellin space [81, 86] to obtain

$$\begin{aligned} B^{\bar\psi_1\psi_2}_{\phi_3\phi_4} &= \langle S_1S_2\rangle \prod_{1\le i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,X_{il}^{-s_{il}}\Gamma(s_{il})\,\mathbb{M}_{2,2}\prod_{i=1}^{4}\hat\delta\left(\Delta_i+\frac{1}{2}(\delta_{i1}+\delta_{i2})-\sum_{j\neq i}s_{ij}\right)\,,\\ &= \frac{\langle S_1S_2\rangle}{\sqrt{X_{12}}}\prod_{1\le i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}}\Gamma(s_{il}+\delta_{1i}\delta_{2l})\,\mathbb{M}_{2,2}\prod_{i=1}^{4}\hat\delta\left(\tau_i-\sum_{j\neq i}s_{ij}\right). \end{aligned} \tag{4.61}$$

The delta function has been normalized as $\hat\delta(x) = 2\pi i\ \delta(x)$ and the measure is given by $(ds_{il}) = \frac{ds_{il}}{2\pi i}$.

The only non-zero component of the Mellin amplitude of the contact interaction is $\mathcal{M}_1 = \mathbb{M}_{2,2}$. Although, we are interested in the four-point contact Witten diagram, in this case it is simple enough to state the general result for the contact Witten diagram with $2n$ fermions and m scalars. $\mathcal{M}_1$ is the only non-zero component $\forall n,m$ and this is given by,

$$\mathcal{M}_1 \;=\; \mathbb{M}_{2n,m} = \frac{\pi^h}{2}\Gamma\left(\frac{1}{2}\sum_{i=1}^{2n+m}\Delta_i+\frac{n}{2}-h\right)\prod_{i=1}^{2n}\left[\frac{C_{\Delta_i+\frac{1}{2}}}{\Gamma(\Delta_i+\frac{1}{2})}\right]\prod_{i=2n+1}^{2n+m}\left[\frac{C_{\Delta_i}}{\Gamma(\Delta_i)}\right], \tag{4.62}$$

Thus the Mellin amplitude for the contact diagram is simply a constant, independent of the Mellin variables. Eq. (4.62) also reflects the fact that for the interaction considered, the three-point function of two fermions and a scalar is parity even in three dimensions.

4.6.3 Fermion-scalar four-point function: scalar exchange

Let us now consider the mixed fermion-scalar four-point Witten diagram with a scalar exchange as shown in fig. 4.2 (on the left). The correlator is given by,

$$\int d^{d+1}z_1\sqrt{g(z_1)}\int d^{d+1}z_2\sqrt{g(z_2)}\;\bar\Sigma_{\Delta_1}(z_1,\vec x_1)\Sigma_{\Delta_2}(z_1,\vec x_2)G_\Delta(z_1,z_2)\prod_{i=3}^{4}K_{\Delta_i}(z_2,\vec x_i). \tag{4.63}$$

Δ is the conformal dimension of the exchanged operator and G_Δ is the scalar bulk-to-bulk propagator. We can simplify eq. (4.63) using eq. (4.58) and the result can be expressed in embedding space language as follows:

$$A^{\bar\psi_1\psi_2}_{\phi_3\phi_4} = \langle S_1S_2\rangle\int_{AdS}dZ_1\int_{AdS}dZ_2\prod_{i=1}^{2}K_{\Delta_i+\frac{1}{2}}(Z_1,X_i)G_\Delta(Z_1,Z_2)\prod_{i=3}^{4}K_{\Delta_i}(Z_2,X_i). \tag{4.64}$$

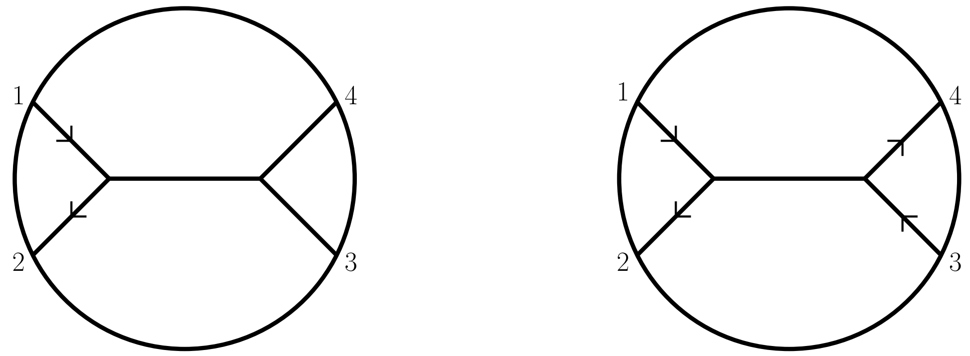


Figure 4.2: Four-point Witten diagrams with scalar exchange.

The correlator in eq. (4.64) is simply given by the corresponding result for the correlator of scalars [81]. This can simply be calculated using eq. (4.60) and expressing the scalar bulk-to-bulk propagator as a convolution of two scalar bulk-to-boundary propagators (see [218, 219]) as follows:

$$\begin{aligned} G_\Delta(Z_1,Z_2) &= \int_{-i\infty}^{i\infty}\frac{d\nu}{2\pi i}\frac{1}{2\pi^{2h}\left[\left(\Delta-\frac{d}{2}\right)^2-\nu^2\right]\Gamma(\nu)\Gamma(-\nu)}\int_{\partial AdS}dQ\int\left[d^2s\right]e^{2(sZ_1+\bar{s}Z_2)\cdot Q}\,, \\ \left[d^2s\right] &\equiv ds\,s^{h+c-1}d\bar{s}\,\bar{s}^{h-c-1}\,. \end{aligned} \tag{4.65}$$

Eventually, the correlator in eq. (4.64) can be expressed in Mellin space as follows:

$$\begin{aligned} A^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} &= \frac{\langle S_1S_2\rangle}{\sqrt{X_{12}}}\prod_{1\leq i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}}\Gamma\left(s_{il}+\delta_{1i}\delta_{2l}\right) \\ &\quad \mathbb{N}^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4}\left(s_{il}\right)\prod_{i=1}^{4}\hat{\delta}\left(\tau_i-\sum_{j\neq i}s_{ij}\right)\,. \end{aligned} \tag{4.66}$$

$\mathcal{M}_1$ is the only non-zero component of the Mellin amplitude and is given as a function of the Mandelstam variable $s=\tau_1+\tau_2-2s_{12}$ as follows:

$$\begin{aligned} \mathcal{M}_1=\mathbb{N}^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4}(s_{il}) &= \frac{\mathbb{M}_{2,2}}{\Gamma\left(\frac{\sum_i\Delta_i}{2}+\frac{1}{2}-h\right)\Gamma\left(\frac{\Delta_1+\Delta_2+1-s}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-s}{2}\right)} \\ &\quad \int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{l(c)l(-c)}{(\Delta-h)^2-c^2}\,, \end{aligned} \tag{4.67}$$

$$l(c) = \frac{\Gamma\left(\frac{h+c-s}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-h+c}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-h+c}{2}\right)}{2\Gamma(c)}\,. \tag{4.68}$$

The poles in eq. (4.67) occur when the contour of the integral is pinched between two colliding poles of the integrand. These poles are at $s=\Delta+2m$ which is exactly as predicted for $\mathcal{M}_1$ in sec. 4.4.1. It may appear that there are other such poles from the integral but these are cancelled by the zeroes in the pre-factor. Like in the case of the Witten diagrams

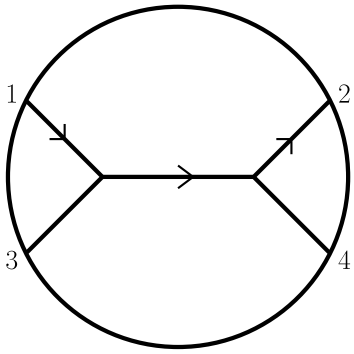


Figure 4.3: Fermion-scalar four-point function: fermion exchange

with scalar external legs [81, 86], the Mellin amplitude can be in fact written as a series over these poles and the residues follow from a simple shift in the corresponding residues from the scalar case.

4.6.4 Fermion-scalar four-point function: fermion exchange

Next, let us look at the Mellin amplitude for the fermion exchange diagram contributing to the mixed fermion-scalar four-point function in fig. 4.3. In [213], it has been shown that the calculation of this Witten diagram can be reduced, although in a slightly more involved way as compared to the previous examples, to that of a scalar exchange diagram [81]. Without going into the details of the calculation (see [2]), let us present the correlator as expressed in Mellin space:

$$\begin{aligned} A^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4} = \; & \frac{\langle S_1 S_2\rangle}{\sqrt{X_{12}}} \prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}} \Gamma\left(s_{il}+\delta_{1i}\delta_{2l}\right) \\ & (\Delta_1+\Delta_3+\Delta+1-d-2s_{13})\, \mathbb{N}^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4}(s_{il}) \prod_{i=1}^{4} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right) \\ & +2\frac{\langle S_1 X_3 X_4 S_2\rangle}{\sqrt{X_{13}X_{34}X_{42}}} \prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}} \bar{\mathbb{N}}^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4}(s_{il}) \\ & \Gamma\left(s_{il}+\frac{1}{2}\left(\delta_{i1}\delta_{l2}+\delta_{i1}\delta_{l3}+\delta_{i3}\delta_{l4}+\delta_{i2}\delta_{l4}\right)\right) \prod_{i=1}^{4} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right) . \end{aligned} \tag{4.69}$$

The Mellin amplitude has two non-zero components $\mathcal{M}_1$ and $\mathcal{M}_2$ which can be expressed as functions of the Mandelstam variable $t = \tau_1 + \tau_3 - 2s_{13}$. $\mathcal{M}_1$ is given by,

$$\begin{aligned}
\mathcal{M}_1 &= (t+\tau+2-d)\,\mathbb{N}^{\bar\psi_1\phi_3}_{\psi_2\phi_4}(s_{il}) \\
&= \frac{(t+\tau+2-d)\,\mathbb{M}_{2,2}}{\Gamma\left(\frac{\sum_i\Delta_i}{2}+\frac{1}{2}-h\right)\Gamma\left(\frac{\tau_1+\tau_3-t}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-t}{2}\right)}\int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{l(c)l(-c)}{(\tau+1-h)^2-c^2}\,, \quad (4.70)\\
l(c) &= \frac{\Gamma\left(\frac{h+c-t-1}{2}\right)\Gamma\left(\frac{\tau_1+\tau_3-h+c+1}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-h+c+1}{2}\right)}{2\Gamma(c)}\,.
\end{aligned}$$

$\mathcal{M}_1$ has poles at $t=\tau+2m$, τ being the twist of the exchanged spinor. Since the three-point function of two spin-half fermions and a scalar is parity even as seen in eq. (4.62), these poles are consistent with our predictions in sec. 4.4.2.

$\mathcal{M}_2$ is given by,

$$\begin{aligned}
\mathcal{M}_2 &= 2\,\bar{\mathbb{N}}^{\bar\psi_1\phi_3}_{\psi_2\phi_4}(s_{il}) \\
&= \frac{2\,\mathbb{M}_{2,2}}{\Gamma\left(\frac{\sum_i\Delta_i}{2}+\frac{1}{2}-h\right)\Gamma\left(\frac{\tau_1+\tau_3-t+1}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-t+1}{2}\right)}\int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{l(c)l(-c)}{(\tau+1-h)^2-c^2}\,, \quad (4.71)\\
l(c) &= \frac{\Gamma\left(\frac{h+c-t}{2}\right)\Gamma\left(\frac{\tau_1+\tau_3-h+c+1}{2}\right)\Gamma\left(\frac{\tau_2+\tau_4-h+c+1}{2}\right)}{2\Gamma(c)}\,.
\end{aligned}$$

$\mathcal{M}_2$ has poles at $t=\tau+1+2m$ which is the precisely the series predicted in sec. 4.4.2 for $\mathcal{M}_2$ when the corresponding three-point functions are parity even.

4.6.5 Fermion four-point function: scalar exchange

Now let us consider the four-point Witten diagram with all external fermionic legs and a scalar exchange as shown in fig. 4.2 (on the right). Following steps to those in the previous examples, we can express this correlator as follows:

$$A^{\bar\psi_1\psi_2}_{\bar\psi_3\psi_4} = \langle S_1S_2\rangle\,\langle S_3S_4\rangle\int_{AdS}dZ_1\int_{AdS}dZ_2\prod_{i=1}^{2}K_{\Delta_i+\frac{1}{2}}(Z_1,X_i)G_\Delta(Z_1,Z_2)\prod_{i=3}^{4}K_{\Delta_i+\frac{1}{2}}(Z_2,X_i). \quad (4.72)$$

The only non-zero component is $\mathcal{M}_1$ which is given by,

$$\begin{aligned}
\mathcal{M}_1 = \mathbb{N}^{\bar\psi_1\psi_2}_{\bar\psi_3\psi_4}(s_{il}) &= \frac{\mathbb{M}_{4,0}}{\Gamma\left(\frac{\sum_i\Delta_i}{2}+1-h\right)\Gamma\left(\frac{\Delta_1+\Delta_2-s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-s}{2}+\frac{1}{2}\right)} \\
&\quad \int_{-i\infty}^{i\infty}\frac{dc}{2\pi i}\frac{l(c)l(-c)}{(\Delta-h)^2-c^2}\,, \quad (4.73)\\
l(c) &= \frac{\Gamma\left(\frac{h+c-s}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-h+c}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-h+c}{2}+\frac{1}{2}\right)}{2\Gamma(c)}\,.
\end{aligned}$$

The poles of $\mathcal{M}_1$ are at $s=\Delta+2m$. In sec. 4.5, we predicted another series of poles at $s=\Delta+1+2m$. One can explain the absence of this second series simply by looking at the parity of the relevant three-point functions. From eq. (4.62), we know that the three-point function of two spin-half fermions and a scalar is parity even and hence the second series of poles (which results from the fusion of the parity even terms of the three-point functions) is absent.

4.7 Mellin amplitudes for conformal Feynman integrals

It has been shown [97–99] in the context of scalar correlators that similar to Witten diagrams, conformal Feynman integrals too can take simple closed form expressions in Mellin space. This has been further established by the Mellin space Feynman rules for tree level conformal integrals with only scalar operators derived in [99]. These diagrammatic rules in Mellin space establish that the Mellin amplitude associated with a tree level conformal Feynman integral with no derivative interaction is simply a product of beta functions, each of which corresponds to an internal propagator in the corresponding Feynman diagram, upto an overall factor of the coupling constants involved. The beta function propagator takes as its argument a Mandestam variable denoting the total Mellin momentum flowing through the propagator, and receives contributions from a series of poles in this Mandelstam variable that correspond to the primary operator exchanged and its descendants.

It is our intention in this section to pursue this direction and calculate Mellin amplitudes associated with tree level conformal Feynman integrals with two or four external fermionic legs. We shall assume that the interactions are Yukawa-like and without derivatives. It is simpler to present these calculations in physical space than in embedding space. The final result shall however be translated to embedding space language so as to make the comparison with the tensor structures in sec. 4.2 more transparent. It is also convenient to assume that these tree level calculations are performed in Euclidean signature. The final result can be Wick rotated implicitly with the correct $i\epsilon$ prescription to Minkowski signature.

The Mellin amplitudes for the conformal Feynman integrals with one or more internal propagators are computed using a recursive method that we describe in detail in appendix. G. In our Feynman diagrams, solid lines with arrows will denote fermion propagators and solid lines without arrows will denote scalar propagators. As in sec. 4.6, the results presented here are independent of the spacetime dimension, however when referring to the parity of correlators and their pole structure, we shall be assuming that we are in three spacetime dimensions.

4.7.1 Fermion-scalar four-point function: contact diagram

Let us first consider the contact interaction with two fermionic and two scalar diagrams as shown in the fig. 4.4 (on the left).

The conformal integral for the contact interaction with two fermionic and two scalar legs

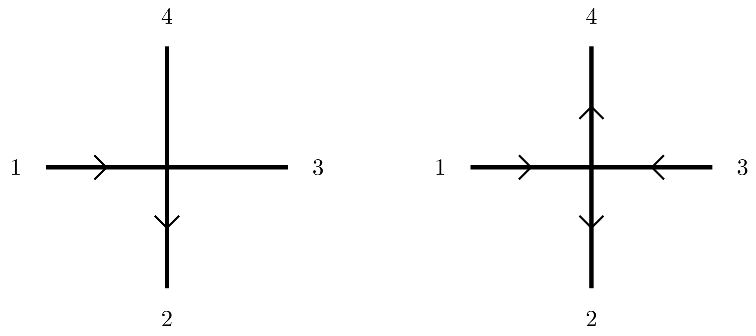


Figure 4.4: Four leg contact interaction.

is given by,

$$C^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} = \int \mathcal{D}u \frac{\not{x}_1 - \not{u}}{|x_1-u|^{2\Delta_1+1}} \Gamma\left(\Delta_1 + \frac{1}{2}\right) \frac{\not{u} - \not{x}_2}{|u-x_2|^{2\Delta_2+1}} \Gamma\left(\Delta_2 + \frac{1}{2}\right) \frac{\Gamma(\Delta_3)}{|x_1-u|^{2\Delta_3}} \frac{\Gamma(\Delta_4)}{|x_4-u|^{2\Delta_4}}, \tag{4.74}$$

where $\mathcal{D}u = \frac{1}{2}\frac{d^d u}{\pi^{d/2}}$.

For eq. (4.74) to be a conformal integral, we must have $\sum \Delta_i = d$. This Mellin representation for this contact interaction was presented by Symanzik [220]. This expression can be expressed using embedding space language in the following manner:

$$\sum_{j=3}^{4} \frac{\langle S_1 X_j S_2\rangle}{\sqrt{X_{1j}X_{j2}}} \prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{l2}} \Gamma\left(s_{il} + \frac{1}{2}(\delta_{i1}+\delta_{i2})\,\delta_{jl} + \frac{1}{2}\delta_{i1}\delta_{l2}\right) \prod_i \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right). \tag{4.75}$$

From eq. (4.75), we can read off the Mellin amplitude associated to this conformal integral. There are two non-zero components, $\mathcal{M}_3$ and $\mathcal{M}_4$, both of which are proportional to 1. Thus the Mellin amplitude corresponding to contact interaction is a constant.

This result can be generalized to incorporate more (or less) number of scalar legs in a straightforward manner. That $\mathcal{M}_3$ and $\mathcal{M}_4$ are the only non-zero components of the Mellin amplitude reflects the fact that the three-point function of two spin one-half fermions and one scalar in this case is parity odd in three dimensions.

4.7.2 Fermion four-point function: contact diagram

Now we shall compute the Mellin amplitude corresponding to the contact diagram with four fermionic legs in fig. 4.4 (on the right). The corresponding conformal integral is given by,

$$
\begin{aligned}
C^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4} &= \int \mathcal{D}u \prod_{i=1}^{4} \Gamma\left(\Delta_i + \frac{1}{2}\right) \left[\frac{\not{x}_1 - \not{u}}{|x_1 - u|^{2\Delta_1+1}} \frac{\not{u} - \not{x}_2}{|u - x_2|^{2\Delta_2+1}}\right] \\
&\qquad \left[\frac{\not{x}_3 - \not{u}}{|x_3 - u|^{2\Delta_3+1}} \frac{\not{u} - \not{x}_4}{|u - x_4|^{2\Delta_4+1}}\right], \qquad (4.76)
\end{aligned}
$$

with the conformality condition $\sum \Delta_i = d$.

In eq. (4.76) all spinor indices are suppressed and square brackets have been used to denote the tensor product. For example, $[\not{x}_1\not{x}_2][\not{y}_1\not{y}_2]$ denotes $(\not{x}_1\not{x}_2)^{\alpha_1}_{\alpha_2}(\not{y}_1\not{y}_2)^{\beta_1}_{\beta_2}$.

The Mellin representation of this conformal integral was also presented in [220].This result can be expressed in embedding space language as follows,

$$
\begin{aligned}
&\prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il})\, X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{l2}-\frac{1}{2}\delta_{i3}\delta_{l4}} \left[\frac{1}{2}\frac{\langle S_1\Gamma^A S_2\rangle \langle S_3\Gamma_A S_4\rangle}{\sqrt{X_{12}X_{34}}}\Gamma\left(s_{il} + \delta_{i1}\delta_{l2} + \delta_{i3}\delta_{l4}\right) + \right. \\
&\left.\sum_{j=3}^{4}\sum_{k=1}^{2} \frac{\langle S_1 X_j S_2\rangle \langle S_3 X_k S_4\rangle}{\sqrt{X_{1j}X_{j2}X_{3k}X_{k4}}}\Gamma\Big(s_{il} + \frac{1}{2}\left(\delta_{i1}+\delta_{i2}\right)\delta_{lj} + \frac{1}{2}\left(\delta_{3l}+\delta_{4l}\right)\delta_{ik} + \frac{1}{2}\delta_{i1}\delta_{l2} + \frac{1}{2}\delta_{i3}\delta_{l4}\Big)\right] \\
&\prod_{i} \hat{\delta}\left(\tau_i - \sum_{j\neq i} s_{ij}\right). \qquad (4.77)
\end{aligned}
$$

The first tensor structure in eq. (4.77) can be expanded in our basis eq. (4.23) as follows,

$$
\frac{\langle S_1\Gamma^A S_2\rangle \langle S_3\Gamma_A S_4\rangle}{\sqrt{X_{12}X_{34}}} = \frac{1}{2}p_1 + 2\sqrt{\frac{v}{u}}p_3 + \frac{1}{2}p_4 - 2\sqrt{\frac{v}{u}}p_5\,. \qquad (4.78)
$$

The physical space tensor corresponding to this 5d invariant is given as follows:

$$
\langle S_1\Gamma^A S_2\rangle \langle S_3\Gamma_A S_4\rangle \xrightarrow[\text{physical space}]{} \left[\not{x}_1\gamma^\mu + \gamma^\mu\not{x}_2\right]\left[\not{x}_3\gamma_\mu + \gamma_\mu\not{x}_4\right] - 2\left[\not{x}_1\not{x}_2\right][\mathbb{I}] - 2[\mathbb{I}]\left[\not{x}_3\not{x}_4\right]. \qquad (4.79)
$$

Applying eq. (4.78) in eq. (4.77), we can read off the Mellin amplitude with respect to our chosen basis. The non-zero components of the Mellin amplitude are the following,

$$
\begin{aligned}
&\mathcal{M}_1 = \frac{1}{4}, \qquad \mathcal{M}_3 = 1, \qquad \mathcal{M}_4 = \frac{1}{4}, \qquad \mathcal{M}_5 = s_{13} - 1, \\
&\mathcal{M}_6 = s_{23}, \qquad \mathcal{M}_7 = s_{14}, \qquad \mathcal{M}_8 = s_{24}\,. \qquad (4.80)
\end{aligned}
$$

4.7.3 Fermion-scalar four-point function: scalar exchange

Let us now calculate the Mellin amplitude corresponding to the exchange of a scalar in the fermion-scalar four-point function as shown in fig. 4.5 (on the left). The conformal integral

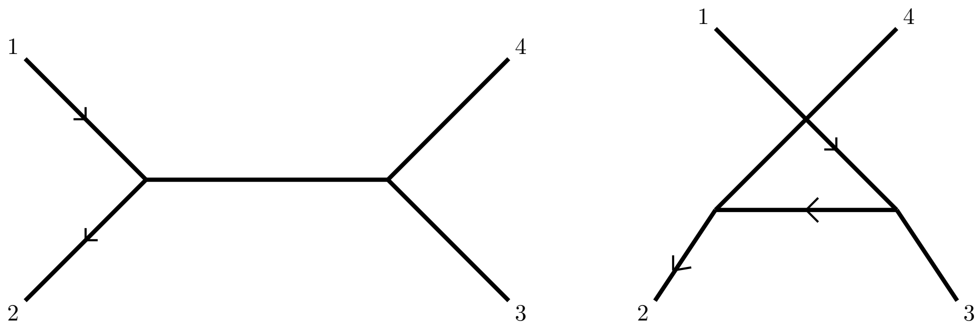


Figure 4.5: Fermion scalar four-point diagrams with scalar and fermionic exchange.

corresponding to the Feynman diagram in fig. 4.5 is given by,

$$\begin{aligned} I^{\bar{\psi}_1\psi_2}_{\phi_3\phi_4} &= \int \mathcal{D}u_1 \int \mathcal{D}u_2 \frac{\not{x}_1 - \not{u}_1}{|x_1-u_1|^{2\Delta_1+1}}\Gamma\left(\Delta_1+\frac{1}{2}\right)\frac{\not{u}_1-\not{x}_2}{|u_1-x_2|^{2\Delta_2+1}}\Gamma\left(\Delta_2+\frac{1}{2}\right) \\ & \prod_{i=3}^{4}\frac{\Gamma(\Delta_i)}{|x_i-u_2|^{2\Delta_i}}\frac{1}{|u_1-u_2|^{2\gamma}}. \end{aligned} \tag{4.81}$$

To ensure the conformality of the integral, we need to impose $\Delta_1+\Delta_2=\Delta_3+\Delta_4=d-\gamma$. The computation of the associated Mellin amplitude is carried out using the recursive method described in appendix. G. We directly state the result here,

$$\begin{aligned} &\sum_{j=3}^{4}\frac{\langle S_1X_jS_2\rangle}{\sqrt{X_{1j}X_{j2}}} \qquad \prod_{1\le i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{l2}}\Gamma\left(s_{il}+\frac{1}{2}(\delta_{1i}+\delta_{2i})\,\delta_{lj}+\frac{1}{2}\delta_{i1}\delta_{l2}\right) \\ &\qquad\qquad \frac{1}{2\Gamma(\gamma)}B\left(\frac{\gamma-s}{2},\frac{d}{2}-\gamma\right)\prod_i\hat{\delta}\left(\tau_i-\sum_{j\neq i}s_{ij}\right). \end{aligned} \tag{4.82}$$

$B(a,b)$ is the beta function. The non-zero components of the Mellin amplitude can now be read off to be,

$$\mathcal{M}_3=\mathcal{M}_4=\frac{1}{2\Gamma(\gamma)}B\left(\frac{\gamma-s}{2},\frac{d}{2}-\gamma\right). \tag{4.83}$$

The poles are located at $-(p_1+p_2)^2=s=\gamma+2m$. This is in agreement with our predictions in sec. 4.4.1. This result can also be generalized trivially to incorporate more scalar legs at either of the two interaction vertices.

4.7.4 Fermion-scalar four-point function: fermion exchange

We wish to calculate the Mellin amplitude associated with the conformal integral as depicted by the diagram in fig. 4.5 (on the right). This features a propagating spin-half fermion in the mixed fermion-scalar four-point function. The relevant conformal integral is,

$$\begin{aligned} I^{\bar{\psi}_1\phi_3}_{\psi_2\phi_4} &= \int \mathcal{D}u_1\int\mathcal{D}u_2\frac{\not{x}_1-\not{u}_1}{|x_1-u_1|^{2\Delta_1+1}}\Gamma\left(\Delta_1+\frac{1}{2}\right)\frac{\not{u}_1-\not{u}_2}{|u_1-u_2|^{2\gamma+1}} \\ & \frac{\not{u}_2-\not{x}_2}{|u_2-x_2|^{2\Delta_2+1}}\Gamma\left(\Delta_2+\frac{1}{2}\right)\frac{\Gamma(\Delta_3)}{|x_3-u_1|^{2\Delta_3}}\frac{\Gamma(\Delta_4)}{|x_4-u_2|^{2\Delta_4}}. \end{aligned} \tag{4.84}$$

The conformality condition forces the sum of the dimensions corresponding to all the legs at each interaction vertex to equal the spacetime dimension. We state the result for the associated Mellin amplitude:

$$\prod_{1\le i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,X_{il}^{-s_{il}-\frac{1}{2}\delta_{1i}\delta_{2l}}\prod_{i}\hat{\delta}\left(\tau_i-\sum_{j\neq i}s_{ij}\right)\Bigg[\frac{\langle S_1X_3X_4S_2\rangle}{\sqrt{X_{13}X_{34}X_{42}}}\frac{1}{2\Gamma(\tau+1)}$$
$$B\left(\frac{\tau-t}{2},\frac{d}{2}-\tau\right)\prod_{1\le i<l}\Gamma\left(s_{il}+\frac{1}{2}\left(\delta_{i1}\delta_{2l}+\delta_{i1}\delta_{l3}+\delta_{i3}\delta_{l4}+\delta_{i2}\delta_{l4}\right)\right)$$
$$-\frac{\langle S_1S_2\rangle}{\sqrt{X_{12}}}\frac{s_{13}}{2\Gamma(\tau+1)}B\left(\frac{\tau+1-t}{2},\frac{d}{2}-\tau\right)\prod_{1\le i<l}\Gamma\left(s_{il}+\delta_{i1}\delta_{l2}\right)\Bigg].\tag{4.85}$$

The two non-zero components of the Mellin amplitude are,

$$\mathcal{M}_1 = -\frac{\tau_1+\tau_3-t}{4\Gamma(\tau+1)}B\left(\frac{\tau+1-t}{2},\frac{d}{2}-\tau\right),\tag{4.86}$$
$$\mathcal{M}_2 = \frac{1}{2\Gamma(\tau+1)}B\left(\frac{\tau-t}{2},\frac{d}{2}-\tau\right).\tag{4.87}$$

$\mathcal{M}_1$ has poles at $-(p_1+p_3)^2=t=\tau+1+2m$ while $\mathcal{M}_2$ has poles at $t=\tau+2m$, where $\tau=\gamma-\frac{1}{2}$ is the twist of the propagating operator. From the results in sec. 4.4.2, we see that these are precisely the series of poles expected when three-point function of two spinors and a scalar is parity odd.

4.7.5 Fermion four-point function: scalar propagator

We shall now calculate the Mellin amplitude for a Feynman diagram with four external fermions and a scalar exchange in the s-channel. The position space conformal integral is given by,

$$\int\mathcal{D}u_1\int\mathcal{D}u_2\left[\frac{\not{x}_1-\not{u}_1}{|x_1-u_1|^{2\Delta_1+1}}\frac{\not{u}_1-\not{x}_2}{|u_1-x_2|^{2\Delta_2+1}}\right]\frac{1}{|u_1-u_2|^{2\gamma}}$$
$$\left[\frac{\not{x}_3-\not{u}_2}{|x_3-u_2|^{2\Delta_3+1}}\frac{\not{u}_2-\not{x}_4}{|u_2-x_4|^{2\Delta_4+1}}\right]\prod_{i=1}^{4}\Gamma\left(\Delta_i+\frac{1}{2}\right).\tag{4.88}$$

The Mellin representation of this integral is given by,

$$I^{\bar{\psi}_1\psi_2}_{\bar{\psi}_3\psi_4} = \prod_{1\le i<l}^{4}\int_{c_{il}-i\infty}^{c_{il}+i\infty}(ds_{il})\,X_{il}^{-s_{il}-\frac{1}{2}\delta_{i1}\delta_{j2}-\frac{1}{2}\delta_{i3}\delta_{j4}}\prod_{i}\hat{\delta}\left(\tau_i-\sum_{j\neq i}s_{ij}\right)$$
$$\Bigg[\frac{1}{2}\frac{\langle S_1\Gamma^AS_2\rangle\langle S_3\Gamma_AS_4\rangle}{\sqrt{X_{12}X_{34}}}\frac{1}{2\Gamma(\gamma)}B\left(\frac{\gamma-s+1}{2},\frac{d}{2}-\gamma\right)\prod_{i<l}\Gamma\left(s_{il}+\delta_{1i}\delta_{2j}+\delta_{3i}\delta_{4j}\right)$$
$$+\sum_{j=3}^{4}\sum_{k=1}^{2}\frac{\langle S_1X_jS_2\rangle\langle S_3X_kS_4\rangle}{\sqrt{X_{1j}X_{j2}X_{3k}X_{k4}}}\frac{1}{2\Gamma(\gamma)}B\left(\frac{\gamma-s}{2},\frac{d}{2}-\gamma\right)$$
$$\prod_{i<l}\Gamma\left(s_{il}+\frac{1}{2}\left(\delta_{1i}+\delta_{2i}\right)\delta_{jl}+\frac{1}{2}\delta_{ik}\left(\delta_{3l}+\delta_{4l}\right)+\frac{1}{2}\left(\delta_{1i}\delta_{2j}+\delta_{3i}\delta_{4j}\right)\right)\Bigg].\tag{4.89}$$

Like in sec. 4.7.2, we have to expand the integral above in our chosen basis of tensor structures eq. (4.23) using eq. (4.78). This gives us the Mellin amplitude in the chosen basis with the following non-zero components:

$$\begin{aligned}
\mathcal{M}_1 &= \mathcal{M}_4 = \frac{1}{8\Gamma(\gamma)} B\left(\frac{\gamma+1-s}{2}, \frac{d}{2}-\gamma\right), \\
\mathcal{M}_3 &= \frac{1}{2\Gamma(\gamma)} B\left(\frac{\gamma-s}{2}, \frac{d}{2}-\gamma\right), \\
\mathcal{M}_5 &= \frac{s_{13}-1}{2\Gamma(\gamma)} B\left(\frac{\gamma-s}{2}, \frac{d}{2}-\gamma\right), \\
\mathcal{M}_6 &= \frac{s_{23}}{2\Gamma(\gamma)} B\left(\frac{\gamma-s}{2}, \frac{d}{2}-\gamma\right), \\
\mathcal{M}_7 &= \frac{s_{14}}{2\Gamma(\gamma)} B\left(\frac{\gamma-s}{2}, \frac{d}{2}-\gamma\right), \\
\mathcal{M}_8 &= \frac{s_{24}}{2\Gamma(\gamma)} B\left(\frac{\gamma-s}{2}, \frac{d}{2}-\gamma\right).
\end{aligned} \tag{4.90}$$

The poles of all the non-zero $\mathcal{M}_i$ above agree with the predictions in sec. 4.5 considering that the relevant three-point functions here are of odd parity.

Chapter 5

AdS/CFT correspondence and the Operator Product Expansion

In the previous two chapters, we have explored two interesting approaches to study CFTs analytically. The first of these was based on the crossing equation using which we showed that the spectrum of every defect CFT has a sector of transverse derivative operators, the OPE data corresponding to which approaches universality for large values of transverse spin s. Furthermore, we could solve the crossing equation perturbatively around this large spin limit to calculate anomalous dimensions and OPE coefficients of transverse derivative operators for finite s. We also derived an inversion formula to the defect channel OPE that established the analyticity of this CFT data as a function of the transverse spin s above a certain threshold. In the second approach, the key ingredient was the Mellin transform which offers a particularly useful representation for conformal correlation functions as the OPE data going into a correlator is made manifest in the analytic structure of the corresponding Mellin amplitude. Although we did not delve into this topic, one of the important applications of the Mellin representation has been to bootstrap CFT data [104, 105] by demanding consistency of the analytic structure of an already crossing symmetric combination of Mellin amplitudes with the OPE. Thus, the primary objective of our explorations has always been to devise methods that help us calculate and understand the properties of CFT data, namely dimensions of operators and the corresponding OPE coefficients.

We shall stay true to this theme and now discuss some properties satisfied by CFT data, OPE coefficients in particular, in the context of the AdS/CFT correspondence. The AdS/CFT correspondence is a conjectured non-perturbative duality between string theories in AdS_{d+1} and conformal field theories living on its d dimensional boundary. In this chapter, we will see that even without an explicit description of the worldsheet CFT in AdS, we can reproduce some basic features of the boundary CFT from the physics on the worldsheet, simply assuming the existence of these dual theories. We shall draw inspiration from [145], where it was shown, in the context of scalars, that the OPE of single trace operators in the boundary CFT can be reproduced from the corresponding OPE of the dual vertex operators in the worldsheet CFT in a theory independent manner. We shall generalize this result to incorporate the contribution of spinning operators to the OPE of scalars and also to the contribution of scalars to the OPE of conserved spin one currents. Furthermore, this shall enable us to derive explicit relations between OPE coefficients corresponding to dual operators in the boundary and worldsheet CFTs. In the supergravity regime of the bulk string theory, the cubic couplings in this bulk theory can be related to the OPE coefficients in the boundary CFT simply by evaluating the three-point Witten diagrams. When the concerned two and three-point functions of the boundary CFT are subject to a non-renormalization theorem, this gives us a complete triangle of relations between the OPE coefficients in the boundary CFT, OPE coefficients in the worldsheet theory and cubic couplings in the bulk supergravity theory.

In sec. 5.1, we shall briefly review the analysis presented in [145] to reproduce the contribution of a scalar to the OPE of two scalar operators in the boundary theory from the corresponding OPE of dual vertex operators in worldsheet CFT. In sec. 5.2, we consider the worldsheet OPE of vertex operators dual to boundary scalars and perform the saddle point analysis and the associated approximations on the contribution of vertex operators dual to spinning operators in the boundary CFT. In sec. 5.3, we shall subject the four-point function of scalars in the boundary theory to the worldsheet OPE analysis based on sec. 5.1 and sec. 5.2 to obtain an explicit relationship between the OPE coefficients of dual integer spin operators in the boundary and worldsheet CFTs. In sec. 5.4, we shall perform a similar analysis focusing on the contribution of a scalar to the OPE of two conserved spin one currents in the boundary theory, thus obtaining the relation satisfied by the corresponding OPE coefficients in the worldsheet and the boundary theories. In sec. 5.5, we note some known results relating boundary OPE coefficients and bulk AdS couplings which shall complete the triangle of relations mentioned above between OPE coefficients in the boundary CFT, OPE coefficients in the bulk worldsheet CFT and cubic couplings in AdS supergravity.

This chapter is based on and contains text from the author's publication [1].

5.1 Inspiration

Our work derives inspiration from the general analysis of [145] which relates the operator product expansions in the boundary CFT and the worldsheet CFT describing the dual bulk string theory. Let us consider Euclidean CFTs in d dimensions which can admit dual descriptions in terms of weakly coupled string theories on $AdS_{d+1} \times M$, where M is a compact manifold. The vertex operators on the worldsheet which create perturbative single string states can be related to a special class of local operators in the dual boundary CFT by the AdS/CFT correspondence as,

$$\mathcal{O}_{\Delta,q}(x) = \int d^2z \; \mathcal{V}_{\Delta,q}(x; z, \bar{z}) \,. \tag{5.1}$$

In eq. (5.1), we leave the relative normalization between worldsheet and boundary CFT operators implicit for now. We shall take this into account explicitly in sec. 5.3. In the above expression, the integration is over the worldsheet coordinates. $\mathcal{V}$ denotes the worldsheet vertex operator, Δ is the scaling dimension of the boundary operator $\mathcal{O}$ and q denotes quantum numbers for additional possible global symmetries of the boundary CFT. The worldsheet scaling dimension of $\mathcal{V}$ will be denoted by $h(\Delta, q)$, h being the sum of the left and right worldsheet conformal dimensions. Following [145], let us use the terminology of large N gauge theories to refer to the operators $\mathcal{O}_{\Delta,q}$ (which create single string states) as "single-trace" operators.

The vertex operators in the physical Hilbert space of the worldsheet CFT are labelled by $\Delta = \frac{d}{2} + 2is$, $s \in \mathbb{R}$. In the boundary CFT, these correspond to the principal series representations of the d-dimensional Euclidean conformal group $SO(d+1,1)$ [167, 206] and by the standard AdS/CFT dictionary, are dual to normalizable modes in the bulk AdS spacetime. The most general form of the scalar contribution to the OPE of two such vertex operators can be written as,

$$\mathcal{V}_{\Delta_1,q_1}(x,z)\mathcal{V}_{\Delta_2,q_2}(0,0) \supset \sum_q \int_C d\Delta \int d^d x' F(z;x,x';\Delta_i,\Delta;q_i,q)\mathcal{V}_{\Delta,q}(x';0) + \cdots . \tag{5.2}$$

The label C in this expression denotes that the Δ integral is along the contour $\frac{d}{2} + 2is$ and the dots denote the contribution from descendants of the worldsheet vertex operator $\mathcal{V}_{\Delta,q}$. In order to avoid cluttering the notation, we shall henceforth denote the external vertex operators simply as $\mathcal{V}_1$ and $\mathcal{V}_2$. The function $F(z;x,x';\Delta_i,\Delta;q_i,q)$ ensures covariance of the OPE with boundary and worldsheet conformal symmetry and contains dynamical data from the worldsheet CFT as well.

We now want to relate the worldsheet OPE in eq. (5.2) to the OPE in the dual boundary CFT. However we are interested in boundary operators belonging to unitary representations of the Lorentzian boundary CFT which are labelled by *real* values of Δ satisfying the unitarity bound $\Delta \geq \frac{d}{2} - 1$ (for scalars) [146]. These operators are dual to non-normalizable modes in AdS and consequently are not present in the physical Hilbert space of states in the worldsheet CFT. Thus, we need to analytically continue the OPE expression in eq. (5.2) to real values of Δ_1, Δ_2 and Δ. Generally, as a result of such analytic continuation, the function $F(z;x,x';\Delta_i,\Delta;q_i,q)$ can develop poles and this will yield additional contributions to the OPE in eq. (5.2). In some cases, these can be shown to be related to the contribution of "multi-trace" operators to the boundary OPEs [121]. In this work, we shall not concern ourselves with the analysis of such "multi-trace" contributions. We proceed with the assumption that apart from these extra contributions, the form of the worldsheet OPE in eq. (5.2) is preserved.

We also note that the operator $\mathcal{V}_{\Delta,q}$ appearing in the OPE eq. (5.2) is a scalar operator from the point of view of the boundary CFT. In general, the worldsheet OPE of scalars will receive contributions from vertex operators which carry boundary Lorentz indices. We shall consider these in the next section.

Using the conformal symmetries of the worldsheet and the boundary theories, we can fix the form of the function $F(z;x,x';\Delta_i,\Delta;q_i,q)$ appearing in eq. (5.2) upto a factor proportional to the relevant OPE coefficient from the worldsheet CFT. Ignoring the contributions of the worldsheet descendant operators, we have,

$$\mathcal{V}_1(x,z)\mathcal{V}_2(0,0) \supset \sum_q \int_C d\Delta \int d^d x' \frac{|z|^{-(h(1)+h(2)-h(\Delta,q))}}{|x|^\alpha |x'|^\beta |x'-x|^\gamma} F^{(0,0,0)}_{12\Delta} \mathcal{V}_{\Delta,q}(x';0)\,, \tag{5.3}$$

where $F^{(0,0,0)}_{12\Delta}$ is the OPE coefficient and this is a function of $\Delta_1, \Delta_2, \Delta, q_1, q_2$ and q. The three zeroes in the superscript denote the values of spin under the boundary Lorentz group of the three operators in the OPE in eq. (5.3).

The form of the OPE in eq. (5.3) and in particular, the parameters α, β, γ can be determined by demanding covariance of the above OPE under conformal transformations on the boundary - see sec. 5.2.1 for details. In particular covariance with scale transformations in the boundary CFT implies:

$$\alpha + \beta + \gamma - d = \Delta_1 + \Delta_2 - \Delta \, . \tag{5.4}$$

Also, the vertex operators which create physical string excitations must be level matched Virasoro primaries with worldsheet conformal dimensions $(1, 1)$. Therefore,

$$h(1) + h(2) = 4 \, . \tag{5.5}$$

In what follows, we shall be interested in analysing the small $|x|$ limit of the OPE in eq. (5.3) since this corresponds to the OPE limit in the boundary CFT. We change coordinates to y such that $x' = y|x|$ and keep only the leading order terms in this limit thus obtaining the following:

$$\mathcal{V}_1(x, z)\mathcal{V}_2(0, 0) \supset \sum_q \int_C d\Delta \frac{|z|^{-(h(1)+h(2)-h(\Delta,q))}}{|x|^{\alpha+\beta+\gamma-d}} F^{(0,0,0)}_{12\Delta} \, \mathcal{V}_{\Delta,q}(0; 0) \int \frac{d^d y}{|y|^\beta |y - \hat{x}|^\gamma} \, , \tag{5.6}$$

where $\hat{x}$ denotes the unit vector corresponding to x.

Consider now the n-point correlation functions of the boundary CFT operators. By the AdS/CFT duality, these can be expressed as correlation functions of the integrated worldsheet vertex operators as,

$$\begin{aligned} \mathcal{A}_n &= \left\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0) \prod_{i=3}^{n-2} \mathcal{O}_i(x_i)\mathcal{O}_{n-1}(x_{n-1})\mathcal{O}_n(x_n) \right\rangle , \\ &= \int d^2 z \left\langle \mathcal{V}_1(x_1; z)\bar{c}c\mathcal{V}_2(0; 0) \prod_{i=3}^{n-2} \int d^2 w_i \mathcal{V}_i(x_i; w_i)\bar{c}c\mathcal{V}_{n-1}(x_{n-1}; 1)\bar{c}c\mathcal{V}_n(x_n; \infty) \right\rangle_{S^2} , \end{aligned} \tag{5.7}$$

where, using the global conformal symmetry on the worldsheet, we have set three of the vertex operator insertions on the worldsheet at $0, 1$ and ∞. The c and $\bar{c}$ are the ghost and anti-ghost insertions which are required to make unintegrated vertex operators BRST invariant. The label S^2 in the worldsheet correlator above implies that we shall consider tree level or genus 0 contributions to the worldsheet correlation functions. In other words, this analysis captures the planar contribution (i.e. large N-limit of the boundary CFT) to the boundary correlation function.

Using the worldsheet OPE in eq. (5.6) and the relations in eq. (5.4) and in eq. (5.5), we obtain on performing the angular worldsheet integration:

$$\mathcal{A}_n \supset |x_1|^{-\Delta_1 - \Delta_2} \sum_q \int d\ln|z| \int_C d\Delta |z|^{h(\Delta,q)-2} |x_1|^\Delta B(\Delta_i, q_i; \Delta, q) \, , \tag{5.8}$$

where,

$$B(\Delta_i, q_i; \Delta, q) = 2\pi F^{(0,0,0)}_{12\Delta} \, \langle \bar{c}c\mathcal{V}_{\Delta,q}(0; 0)X \rangle \int \frac{d^d y}{|y|^\beta |y - \hat{x}_1|^\gamma} \, . \tag{5.9}$$

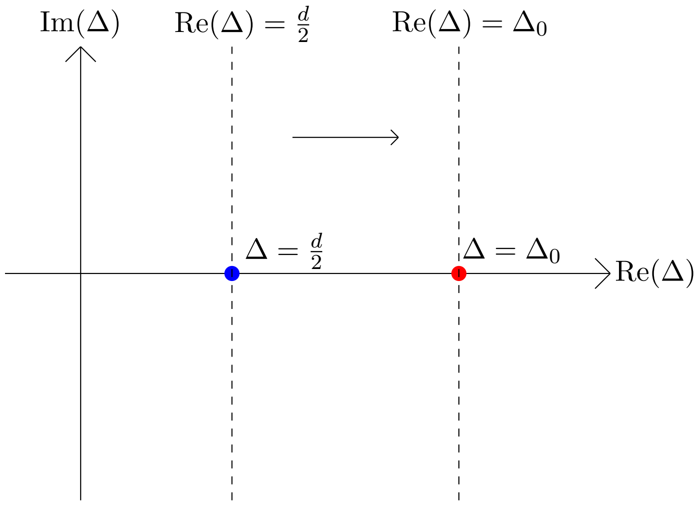


Figure 5.1: The vertical dashed line labelled by $\mathrm{Re}(\Delta) = \frac{d}{2}$ denotes the contour of integration in the OPE of normalizable vertex operators. The original contour is deformed along the real axis as indicated by the arrow such that it intersects the point $\Delta = \Delta_0$ (marked in red), which is the saddle point of the Δ integral in eq. (5.8).

In the above expression, X includes all the $n-3$ integrated vertex operator insertions at w_i and the two fixed vertex operators at 1 and ∞ of eq. (5.7).

Although not exact, the expression in eq. (5.8) yields the dominant contribution to the correlator in the small $|z|$ and small $|x_1|$ limit. The Δ-integral in eq. (5.8) can be shown to admit a saddle point at $\Delta = \Delta_0$ such that,

$$\partial_\Delta h(\Delta, q)|_{\Delta=\Delta_0} = -\frac{\ln|x_1|}{\ln|z|}\,.$$

It turns out that the values of Δ_0 satisfying this condition, in general, do not lie on the original contour of integration. Assuming the integrand to be an analytic function of Δ, (ignoring the possibility of poles on the complex Δ plane), we can deform the integration contour such that it intersects the real Δ-axis at Δ_0 as illustrated in fig. 5.1. As discussed in [145], the poles which we may cross while shifting the contour, do not change the conclusion of this analysis.

Having performed the Δ integral via saddle point, we can again resort to a saddle point approximation for evaluating the z-integral. The saddle point condition for the z integral gives

$$h(\Delta_0, q) = 2\,.$$

This is precisely the condition which must hold for a worldsheet vertex operator to correspond to a physical string state in the bulk AdS target space. Consequently (via the AdS/CFT duality) this corresponds to a single trace operator in the boundary CFT.

To ensure that the saddle approximation is justified, we also need to check that the higher order fluctuations around the saddle value are suppressed. This indeed turns out to be the case provided $\partial_\Delta h|_{\Delta_0} < 0$ and $\partial^2_\Delta h|_{\Delta_0} < 0$. A straightforward computation of the Gaussian fluctuations around the saddle point finally gives the dominant contribution to the integral in the spacetime OPE limit $|x_1| \to 0$ as,

$$\mathcal{A}_n \supset -2i\pi \sum_q \frac{|x_1|^{\Delta_0-\Delta_1-\Delta_2}}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}} B(\Delta_i, q_i; \Delta_0, q)\,. \tag{5.10}$$

The dependence of the above expression on $|x_1|$ and the $(n-1)$-point correlator in the boundary CFT through the $(n-1)$-point worldsheet correlator in $B(\Delta_i, q_i; \Delta_0, q)$, is exactly the form we expect when we consider the contribution to the boundary n-point function from the exchange of a single trace operator of dimension Δ_0 in the OPE of $\mathcal{O}_{\Delta_1}$ and $\mathcal{O}_{\Delta_2}$.

Up till now, we have reviewed the analysis in [145] where we have considered the contribution of only scalar operators in the OPE. In our work, we shall generalize this to include the contribution of operators with spin. Furthermore, we shall apply this analysis to extract concrete relations between the OPE coefficients in the worldsheet and boundary theories by expressing eq. (5.10) for four-point functions in a factorized form in the (12)(34) channel. The saddle point analysis involved in our work is identical to that described above and hence we shall not present the details of such calculations any further.

5.2 Worldsheet OPE of scalars

In this section, we generalize the OPE of scalar operators in eq. (5.2) to include the contributions of the worldsheet operators which are dual to boundary CFT operators with spin. Conformal symmetry fixes the constants α, β and γ and the tensor structures in such OPE completely.

5.2.1 Structure of the OPE and shadow operators

Let us start with a connection between shadow operators and the structure of the OPE in eq. (5.2) and using this to determine the boundary coordinate dependence in the OPE in eq. (5.2). The z dependence of the function $F(z; x, x'; \Delta_i, \Delta; q_i, q)$ in eq. (5.2) is fixed by worldsheet conformal invariance to be $|z|^{-(h(1)+h(2)-h(\Delta,q))}$. Thus, we can write (working with generic x_i and z_i for the moment),

$$\mathcal{V}_1(x_1,z_1)\mathcal{V}_2(x_2,z_2) \supset \sum_q \int_C d\Delta \int d^dx |z_{12}|^{-(h(1)+h(2)-h(\Delta,q))} F^{(0,0,0)}_{12\Delta} L(x_1,x_2;x)\mathcal{V}_{\Delta,q}(x,z_2)\,. \tag{5.11}$$

In eq. (5.11), the functional dependence of $F(z_i; x_i, x; \Delta_i, \Delta; q_i, q)$ on x_1, x_2 and x is captured by the function $L(x_1, x_2; x)$. $L(x_1, x_2; x)$ is an entirely kinematic factor and we shall now show that the shadow operator formalism provides an efficient way of determining $L(x_1, x_2; x)$.

In a d-dimensional CFT, the shadow of a conformal primary operator $\mathcal{O}^{\mu_1\cdots\mu_l}_\Delta$ of spin l and scaling dimension Δ can be expressed as [175],

$$\tilde{\mathcal{O}}^{\mu_1\cdots\mu_l}_{d-\Delta}(x) \equiv \frac{k_{\Delta,l}}{\pi^{d/2}} \int d^dy\ \frac{1}{|x-y|^{2(d-\Delta)}} J^{\mu_1}_{\ (\nu_1} \cdots J^{\mu_l}_{\ \nu_l)}(x-y)\mathcal{O}^{\nu_1\cdots\nu_l}_\Delta(y)\,, \tag{5.12}$$

where,

$$J^{\mu\nu}(x) = \delta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2}\,, \tag{5.13}$$

and the brackets in the subscript of $J^{\mu_1}_{(\nu_1} \cdots J^{\mu_l}_{\nu_l)}$ denote total symmetrization in those indices. The following normalisation factor

$$k_{\Delta,l} \equiv \frac{1}{(\Delta-1)_l}\frac{\Gamma(d-\Delta+l)}{\Gamma(\frac{2\Delta-d}{2})}\,, \tag{5.14}$$

ensures that

$$\tilde{\tilde{\mathcal{O}}}^{(l)}_{\Delta,\,\mu_1\cdots\mu_l} = \mathcal{O}^{(l)}_{\Delta,\,\mu_1\cdots\mu_l}\,. \tag{5.15}$$

Now if we perform a conformal transformation in the boundary CFT, a scalar primary operator will transform as,

$$\mathcal{O}_{\Delta,q}(x) \to [\Omega(x)]^{\Delta}\,\mathcal{O}_{\Delta,q}(x)\,, \tag{5.16}$$

where $\Omega(x)$ is the conformal factor. This implies that under the boundary conformal transformation, the dual worldsheet vertex operator $\mathcal{V}_{\Delta,q}$ transforms as

$$\mathcal{V}_{\Delta,q}(z,x) \to [\Omega(x)]^{\Delta}\,\mathcal{V}_{\Delta,q}(z,x)\,. \tag{5.17}$$

Since the boundary conformal symmetry acts as a global symmetry on the worldsheet, eq. (5.11) should transform in a covariant fashion when a conformal transformation is implemented in the boundary CFT. Hence, using eq. (5.17) in eq. (5.11) we get,

$$L(x_1,x_2;x) \to [\Omega(x_1)]^{\Delta_1}[\Omega(x_2)]^{\Delta_2}[\Omega(x)]^{d-\Delta}L(x_1,x_2;x)\,. \tag{5.18}$$

But this is precisely how a three-point function of the boundary operators $\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}$ and the shadow operator $\tilde{\mathcal{O}}_{d-\Delta}$ transforms under a conformal transformation. Thus, we have:

$$L(x_1,x_2;x) \propto \left\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\tilde{\mathcal{O}}_{d-\Delta}(x)\right\rangle\,. \tag{5.19}$$

Using eq. (5.31) and eq. (J.2), this three-point function can be computed as [175],

$$\begin{aligned}
&\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\tilde{\mathcal{O}}_{d-\Delta}(x)\rangle \\
=\;& \frac{k_{\Delta,0}}{\pi^{d/2}}\int d^dy\;\frac{1}{|x-y|^{2(d-\Delta)}}\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta}(y)\rangle\,, \\
=\;& \frac{k_{\Delta,0}}{\pi^{d/2}}\int d^dy\;\frac{1}{|x-y|^{2(d-\Delta)}}\left[\frac{\bar{\lambda}^{(0,0,0)}_{\Delta_1\Delta_2\Delta}}{|y-x_1|^{\Delta+\Delta_1-\Delta_2}|y-x_2|^{\Delta+\Delta_2-\Delta_1}|x_1-x_2|^{\Delta_1+\Delta_2-\Delta}}\right]\,, \\
=\;& \frac{H(\Delta_1,\Delta_2,\Delta,d)}{|x_1-x_2|^{\Delta_1+\Delta_2+\Delta-d}|x_2-x|^{\Delta_2-\Delta_1-\Delta+d}|x-x_1|^{\Delta_1-\Delta_2-\Delta+d}}\,,
\end{aligned} \tag{5.20}$$

where, $\bar{\lambda}^{(0,0,0)}_{\Delta_1\Delta_2\Delta}$ is the three-point function coefficient appearing in $\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta}(y)\rangle$ and

$$H(\Delta_1,\Delta_2,\Delta,d) \equiv k_{\Delta,0}\bar{\lambda}^{(0,0,0)}_{\Delta_1\Delta_2\Delta}\frac{\Gamma\left(\frac{\Delta_1-\Delta_2-\Delta+d}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1-\Delta+d}{2}\right)\Gamma\left(\frac{2\Delta-d}{2}\right)}{\Gamma\left(\frac{\Delta+\Delta_1-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta+\Delta_2-\Delta_1}{2}\right)\Gamma\left(d-\Delta\right)}\,. \tag{5.21}$$

We have taken care to denote three-point function coefficients with $\bar{\lambda}_{ijk}^{(l_1,l_2,l_3)}$ to distinguish it from OPE coefficients which we denote with $\lambda_{ijk}^{(l_1,l_2,l_3)}$ because for now we shall not normalize the two-point function to have unit coefficient if the operators have spin. We shall use $\lambda_{ijk}^{(l_1,l_2,l_3)}$ and $\lambda_{\Delta_i\Delta_j\Delta_k}^{(l_1,l_2,l_3)}$ interchangably and similarly for the three-point function coefficient $\bar{\lambda}_{ijk}^{(l_1,l_2,l_3)}$.

Since we are only interested in obtaining the dependence of $L(x_1, x_2; x)$ on x_1, x_2 and x, we choose $(H(\Delta_1, \Delta_2, \Delta, d))^{-1}$ to be the proportionality constant in eq. (5.19). This is justified since eq. (5.19) is a purely kinematical relation which follows from the fact that the boundary conformal symmetry manifests itself as a global symmetry of the worldsheet theory. In other words there is no dynamical information about the worldsheet theory in eq. (5.19). Thus using eq. (5.20) and eq. (5.19) in eq. (5.11) we arrive at the OPE,

$$\mathcal{V}_1(x_1, z_1)\mathcal{V}_2(x_2, z_2) \supset \sum_q \int_C d\Delta \int d^d x' \frac{|z_{12}|^{-(h(1)+h(2)-h(\Delta,q))}}{|x_{12}|^{\alpha}|x_2 - x|^{\beta}|x - x_1|^{\gamma}} F_{12\Delta}^{(0,0,0)}\, \mathcal{V}_{\Delta,q}(x; z_2)\,, \quad (5.22)$$

which is the same as eq. (5.3) if we put $z_1 = z, z_2 = 0, x_1 = x, x_2 = 0$ and $x = x'$ above. Moreover, the values of α, β and γ are fixed to be:

$$\begin{aligned} \alpha &= \Delta_1 + \Delta_2 + \Delta - d\,, \\ \beta &= \Delta_2 - \Delta_1 - \Delta + d\,, \\ \gamma &= \Delta_1 - \Delta_2 - \Delta + d\,. \end{aligned} \quad (5.23)$$

5.2.2 Worldsheet OPE

We now consider the contributions of the vertex operators of the form $\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_l}(x, z)$ to the OPE of two worldsheet scalar vertex operators $\mathcal{V}_1(x_1; z_1)$ and $\mathcal{V}_2(x_2; z_2)$. The additional boundary Lorentz indices $(\mu_1, \cdots \mu_l)$ imply that the corresponding dual operator in the boundary CFT has spin l. From the perspective of the worldsheet theory, these can be interpreted as appropriate global symmetry labels.

The most general form of the worldsheet OPE involving the exchange of such operators can be written as (ignoring the contribution of descendants),

$$\begin{aligned} &\mathcal{V}_1(x_1, z_1)\mathcal{V}_2(x_2, z_2) \\ \supset &\sum_q \int_C d\Delta \int d^d x\ F_{\mu_1\cdots\mu_l}(x_i, x; z_i; \Delta_i, \Delta; q_i, q)\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_l}(x, z_1)\,, \\ = &\sum_q \int_C d\Delta \int d^d x\ |z_{12}|^{-(h(1)+h(2)-h(\Delta,q))} F_{12\Delta}^{(0,0,l)} L_{\mu_1\cdots\mu_l}(x_1, x_2, x)\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_l}(x, z_1)\,. \end{aligned} \quad (5.24)$$

The worldsheet conformal symmetry of the worldsheet theory has been used to fix the z dependence in eq. (5.24). The function $L_{\mu_1\cdots\mu_l}(x_1, x_2, x)$ is just a kinematical factor fixed by the boundary conformal symmetry. In order to obtain the functional form of $L_{\mu_1\cdots\mu_l}(x_1, x_2, x)$ we shall again make use of the shadow operator formalism. Let us consider applying an inversion on the boundary coordinates. This gives,

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \quad , \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2} \quad , \quad d^d x \to \frac{d^d x}{|x|^{2d}} \quad . \tag{5.25}$$

A boundary CFT operator $\mathcal{O}_{\Delta,q}^{\mu_1\cdots\mu_l}$ with spin l and dimension Δ transforms under inversion as,

$$\mathcal{O}_{\Delta,q}^{\mu_1\cdots\mu_l}(x) \to |x|^{2\Delta} J^{\mu_1}_{\nu_1}(x)\cdots J^{\mu_l}_{\nu_l}(x)\, \mathcal{O}_{\Delta,q}^{\nu_1\cdots\nu_l}(x)\,, \tag{5.26}$$

with $J^{\mu_1}_{\nu_1}(x)$ given by eq. (5.13).

The dual worldsheet operator $\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_l}(x,z)$ then tranforms under boundary conformal transformation as,

$$\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_l}(x,z) \to |x|^{2\Delta} J^{\mu_1}_{\nu_1}(x)\cdots J^{\mu_l}_{\nu_l}(x)\, \mathcal{V}_{\Delta,q}^{\nu_1\cdots\nu_l}(x,z)\,. \tag{5.27}$$

For $l = 0$ we will have

$$\mathcal{O}_{\Delta,q}(x) \to |x|^{2\Delta}\mathcal{O}_{\Delta,q}(x), \qquad \mathcal{V}_{\Delta,q}(x,z) \to |x|^{2\Delta}\mathcal{V}_{\Delta,q}(x,z)\,. \tag{5.28}$$

Using these relations it is evident that to preserve the form of the OPE in eq. (5.24) under inversion, $L_{\mu_1\cdots\mu_l}(x_1,x_2,x)$ must transform as,

$$L_{\mu_1\cdots\mu_l}(x_1,x_2,x) \to |x_1|^{2\Delta_1}|x_2|^{2\Delta_2}|x|^{2(d-\Delta)}\left(J^{\mu_1}_{\nu_1}(x)\cdots J^{\mu_l}_{\nu_l}(x)\right)^{-1} L_{\nu_1\cdots\nu_l}(x_1,x_2,x)\,, \tag{5.29}$$

where, the inverse of $J^{\mu\nu}$ in the above expression is given by the identity $J^{\mu\rho}(x)J_{\rho\nu}(x) = \delta^\mu{}_\nu$.

It is easy to verify that a three-point function of $\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}$ and $\tilde{\mathcal{O}}_{d-\Delta}^{\mu_1\cdots\mu_l}$ transforms exactly in the above fashion under inversion, which implies that,

$$L_{\mu_1\cdots\mu_l}(x_1,x_2,x) \propto \left\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\tilde{\mathcal{O}}_{d-\Delta}^{\mu_1\cdots\mu_l}(x)\right\rangle\,. \tag{5.30}$$

The three-point function of two scalars and one spin l operator is given by,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3^{\mu_1\cdots\mu_l}(x_3)\rangle = \frac{\bar{\lambda}_{123}^{(0,0,l)}\Big(V^{\mu_1}(x_1,x_2,x_3)\cdots V^{\mu_l}(x_1,x_2,x_3) - \text{traces}\Big)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3+l}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1-l}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2-l}}\,, \tag{5.31}$$

where,

$$V^\mu(x_1,x_2,x_3) = \frac{(x_1-x_3)^\mu}{(x_1-x_3)^2} - \frac{(x_2-x_3)^\mu}{(x_2-x_3)^2}\,. \tag{5.32}$$

Therefore the three-point function in eq. (5.30) can be obtained using eq. (5.31), the expression for shadow operators in eq. (5.12) and the integral in eq. (J.3) as [175],

$$\begin{aligned}
&\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\tilde{\mathcal{O}}_{d-\Delta}^{\mu_1\cdots\mu_l}(x)\rangle \\
&= \frac{k_{\Delta,l}}{\pi^{d/2}}\int d^d y\; \frac{J^{\mu_1}_{(\nu_1}\cdots J^{\mu_l}_{\nu_l)}(x-y)}{|x-y|^{2(d-\Delta)}}\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta}^{\nu_1\cdots\nu_l}(y)\rangle\,, \\
&= \frac{k_{\Delta,l}}{\pi^{d/2}}\int d^d y\; \frac{J^{\mu_1}_{(\nu_1}\cdots J^{\mu_l}_{\nu_l)}}{|x-y|^{2(d-\Delta)}}\frac{\bar{\lambda}^{(0,0,l)}_{\Delta_1\Delta_2\Delta}V^{\nu_1}(x_1,x_2,y)\cdots V^{\nu_l}(x_1,x_2,y)}{|y-x_1|^{\Delta+\Delta_1-\Delta_2-l}|y-x_2|^{\Delta+\Delta_2-\Delta_1-l}|x_1-x_2|^{\Delta_1+\Delta_2-\Delta+l}}\,, \\
&= \frac{H'(\Delta_1,\Delta_2,\Delta,d,l)\quad V^{\mu_1}(x_1,x_2,x)\cdots V^{\mu_l}(x_1,x_2,x)}{|x_1-x_2|^{\Delta_1+\Delta_2+\Delta-d+l}|x_2-x|^{\Delta_2-\Delta_1-\Delta+d-l}|x-x_1|^{\Delta_1-\Delta_2-\Delta+d-l}}\,,
\end{aligned} \tag{5.33}$$

where,

$$H'(\Delta_1, \Delta_2, \Delta, d, l) = k_{\Delta,l}\bar{\lambda}^{(0,0,l)}_{\Delta_1\Delta_2\Delta}\frac{\Gamma\left(\frac{\Delta_1-\Delta_2-\Delta+d+l}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1-\Delta+d+l}{2}\right)\Gamma\left(\frac{2\Delta-d}{2}\right)}{\Gamma\left(\frac{\Delta+\Delta_1-\Delta_2+l}{2}\right)\Gamma\left(\frac{\Delta+\Delta_2-\Delta_1+l}{2}\right)\Gamma\left(d-\Delta+l\right)}$$
$$\Lambda_l(\Delta+\Delta_1-\Delta_2-l, \Delta+\Delta_2-\Delta_1-l)\,, \quad (5.34)$$

and,

$$\Lambda_l(a,b) = \prod_{r=0}^{l-1}\left(\frac{a+b}{2}+l-1+r\right). \quad (5.35)$$

Once again, the object of interest is just the functional dependence of $L_{\mu_1\cdots\mu_l}(x_1, x_2, x)$ on the boundary coordinates. Thus, we take the proportionality factor in eq. (5.30) to be $(H'(\Delta_1, \Delta_2, \Delta, d, l))^{-1}$. The contribution to the worldsheet OPE from the exchange of vertex operators dual to boundary spin l operators can then be expressed as,

$$\mathcal{V}_1(x_1, z_1)\mathcal{V}_2(x_2, z_2)$$
$$\supset \sum_q \int_C d\Delta \int d^dx\, \frac{|z_{12}|^{-(h_1+h_2-h_{\Delta,q})}}{|x_1-x_2|^{\alpha}|x_2-x|^{\beta}|x-x_1|^{\gamma}} \frac{V_{\mu_1}\cdots V_{\mu_l}}{} F^{(0,0,l)}_{12\Delta}\mathcal{V}^{\mu_1\cdots\mu_l}_{\Delta,q}(x, z_1)\,, \quad (5.36)$$

where α, β, γ in the above expression are given by

$$\begin{aligned} \alpha &= \Delta_1+\Delta_2+\Delta-d+l\,, \\ \beta &= \Delta_2-\Delta_1-\Delta+d-l\,, \\ \gamma &= \Delta_1-\Delta_2-\Delta+d-l\,. \end{aligned} \quad (5.37)$$

Thus, we see that the dependence of the worldsheet OPE on the coordinates of the boundary spacetime can be rather efficiently obtained by appealing to the notion of shadow operators. In sec. 5.3, we shall show that the tensor structures in the worldsheet OPE obtained via this method indeed reproduces the tensor structures that appear in the OPE of the dual operators in the boundary CFT theory.

We can simplify the expression in eq. (5.36) by putting the vertex operator $\mathcal{V}_2$ at the origin of the x and z coordinate systems and change the integration variable to y such that $x = y|x_1|$. We can then expanding around $x_1 = 0$ and keep only the leading terms (since we are ignoring the contributions of descendants) to obtain the following,

$$\mathcal{V}_1(x_1, z_1)\mathcal{V}_2(0,0) \;\supset\; \sum_q \int_C d\Delta \frac{|z_1|^{-(h_1+h_2-h_{\Delta,q})}}{|x_1|^{\alpha+\beta+\gamma+l-d}} F^{(0,0,l)}_{12\Delta}\mathcal{V}^{\mu_1\cdots\mu_l}_{\Delta,q}(0,0)G_{\mu_1\cdots\mu_l}(x_1)\,, \quad (5.38)$$

where, using the integral in eq. (J.4), we have:

$$\begin{aligned} G^{\mu_1\cdots\mu_l}(x_1) &= \int d^dy\, \frac{V^{\mu_1}(\hat{x}_1,0,y)\cdots V^{\mu_l}(\hat{x}_1,0,y)}{|y|^{\beta}\,|y-\hat{x}_1|^{\gamma}}\,, \\ &= \pi^{d/2}\Lambda_l(\beta,\gamma)\frac{\Gamma\left(\frac{d-\beta}{2}\right)\Gamma\left(\frac{d-\gamma}{2}\right)\Gamma\left(\frac{\beta+\gamma-d}{2}+l\right)}{\Gamma\left(\frac{\beta}{2}+l\right)\Gamma\left(\frac{\gamma}{2}+l\right)\Gamma\left(\frac{2d-\beta-\gamma}{2}\right)}\left(\hat{x}_1^{\mu_1}\cdots\hat{x}_1^{\mu_l}\right). \end{aligned} \quad (5.39)$$

β, γ is given by eq. (5.37) and $\hat{x}_1^{\mu}$ denotes the unit vector corresponding to x_1^{μ}.

5.3 OPE coefficients in the worldsheet and boundary CFTs

Suppose $\lambda_{12\Delta}^{(0,0,l)}$ be the OPE coefficient which appears in the symmetric traceless spin l contribution to the OPE of two boundary scalar operators. Our goal now is to obtain the relationship between $\lambda_{12\Delta}^{(0,0,l)}$ and the worldsheet OPE coefficient $F_{12\Delta}^{(0,0,l)}$. To do this, we shall make use of the saddle point analysis reviewed in sec. 5.1.

Let us consider a four-point correlator of scalar operators in the boundary CFT. We can evaluate this correlator directly in the boundary theory in the OPE limit. Alternatively, we can use the representation of the boundary operators as integrated worldsheet vertex operators and use the saddle point analysis to evaluate it using the worldsheet theory. As mentioned in sec. 5.1, the saddle point analysis gives the result in the OPE limit. Comparing the results from the OPE analysis in the boundary and the worldsheet theories will give us the desired relationship between the OPE coefficients.

Since we intend to deal with relations between OPE coefficients across two theories, we need to fix the relative normalization between the vertex operators in the worldsheet theory and the operators in the boundary CFT. We can do this by comparing the two-point function in the two theories. Choosing the coefficient of two-point functions in the boundary CFT automatically fixes the normalization of the worldsheet vertex operators. This is done in appendix. I. Below, we shall work with the normalized vertex operators.

Let us first look at the OPE in the boundary theory directly. The four-point function of generic scalar operators in the boundary theory can be easily evaluated in the OPE limit. Using the standard OPE between two scalars and the three-point function in eq. (5.31), we obtain:

$$\begin{aligned}
&\Big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\Big\rangle \\
\approx\ & \sum_{\Delta,q} \lambda_{12\Delta}^{(0,0,l)} \frac{(x_1)_{\mu_1}\cdots(x_1)_{\mu_l}}{|x_1|^{(\Delta_1+\Delta_2-\Delta)+l}} \Big\langle \mathcal{O}_{\Delta,q}^{\mu_1\cdots\mu_l}(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\Big\rangle\,, \\
=\ & \sum_{\Delta,q} \frac{(\hat{x}_1)_{\mu_1}\cdots(\hat{x}_1)_{\mu_l}}{|x_1|^{\Delta_1+\Delta_2-\Delta}} \frac{\lambda_{12\Delta}^{(0,0,l)}\bar{\lambda}_{34\Delta}^{(0,0,l)}\Big(V^{\mu_1}(x_3,x_4,0)\cdots V^{\mu_l}(x_3,x_4,0)-\text{Traces}\Big)}{|x_3|^{\Delta_3+\Delta-\Delta_4-l}|x_4|^{\Delta_4+\Delta-\Delta_3-l}|x_{34}|^{\Delta_3+\Delta_4-\Delta+l}}\,,
\end{aligned} \tag{5.40}$$

where q denotes the quantum numbers due to additional global symmetries.

Let $K_l(\Delta)$ denote the two-point function coefficient of two spin l operators with conformal dimension Δ as follows:

$$\langle \mathcal{O}^{\mu_1\cdots\mu_l}(x)\mathcal{O}_{\nu_1\cdots\nu_l}(0)\rangle = K_l(\Delta)\left[\frac{J^{\mu_1}_{(\nu_1}(x)\cdots J^{\mu_l}_{\nu_l)}(x)-\text{Traces}}{|x|^{2\Delta}}\right]. \tag{5.41}$$

$J_{\mu\nu}(x)$ is defined in eq. (5.13). We shall normalise the scalar operators such that $K_0(\Delta)=1$. However, for $l\neq 0$, we shall keep the explicit factors of $K_l(\Delta)$. Thus we shall not need to distinguish between the OPE coefficients $\lambda_{123}^{(0,0,0)}$ and the corresponding three-point function coefficient $\bar{\lambda}_{123}^{(0,0,0)}$.

For generic l, the relation between the OPE coefficient $\lambda_{123}^{(0,0,l)}$ and the three-point function coefficient $\bar{\lambda}_{123}^{(0,0,l)}$ is given by :

$$\bar{\lambda}_{123}^{(0,0,l)} = K_l(\Delta_3)\lambda_{123}^{(0,0,l)} . \tag{5.42}$$

Using eq. (5.42) in eq. (5.40), the contribution of a spin l primary (in the boundary CFT) to the four-point function can be expressed as:

$$\begin{aligned} &\Big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\Big\rangle \\ \approx &\sum_{\Delta,q} \frac{(\hat{x}_1)_{\mu_1}\cdots(\hat{x}_1)_{\mu_l}}{|x_1|^{\Delta_1+\Delta_2-\Delta}} \frac{K_l(\Delta)\lambda_{12\Delta}^{(0,0,l)}\lambda_{34\Delta}^{(0,0,l)}\Big(V^{\mu_1}(x_3,x_4,0)\cdots V^{\mu_l}(x_3,x_4,0) - \text{Traces}\Big)}{|x_3|^{\Delta_3+\Delta-\Delta_4-l}|x_4|^{\Delta_4+\Delta-\Delta_3-l}|x_{34}|^{\Delta_3+\Delta_4-\Delta+l}} . \end{aligned} \tag{5.43}$$

Next, let us subject the same four-point function in the OPE limit to the worldsheet OPE analysis. Unlike in sec. 5.1, we shall now consider the contribution of symmetric traceless spin l operators in the OPE. Following the procedure of sec. 5.1, the four-point function is given in the OPE limit $x_1 \to 0$ as follows,

$$\begin{aligned} &\Big\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\Big\rangle \\ &= \frac{1}{\prod_{i=1}^4 a_i^{(0)}} \int d^2z \langle \mathcal{V}_1(x_1;z)\bar{c}c\mathcal{V}_2(x_2;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\rangle , \\ &\supset -\frac{2i\pi}{\prod_{i=1}^4 a_i^{(0)}} \sum_q \frac{|x_1|^{\Delta_0-\Delta_1-\Delta_2}}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}} \bar{B}_l(\Delta_i,q_i;\Delta_0,q) . \end{aligned} \tag{5.44}$$

Here we have introduced the factors $a_i^{(j)}$ taking into account the relative normalization between $\mathcal{O}_i$ and $\mathcal{V}_i$, the superscript j indicating the boundary spin of the operators involved. Their precise form is given in the appendix. I. $\bar{B}_l$ is given by an expression similar to the one given in eq. (5.9) but now generalized to include the contribution from tensor operators:

$$\bar{B}_l = 2\pi F_{12\Delta_0}^{(0,0,l)} \Big\langle \bar{c}c\mathcal{V}_{\Delta_0,q}^{\mu_1\cdots\mu_l}(0;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\Big\rangle G_{\mu_1\cdots\mu_l}(x_1) . \tag{5.45}$$

$G^{\mu_1\cdots\mu_l}$ is given in eq. (5.39) and the worldsheet OPE coefficient $F_{12\Delta_0}^{(0,0,l)}$ is related to the worldsheet two and three-point functions as shown in appendix. H. Note that it is important to distinguish between the OPE coefficient $F_{12\Delta_0}^{(0,0,l)}$ and the three-point function coefficient $\bar{F}_{12\Delta_0}^{(0,0,l)}$ since they are related by a non trivial factor given in eq. (H.5).

Now we want to calculate $\bar{B}_l$ explicitly. The three-point correlator appearing in eq. (5.45) can be written as the product of a ghost correlator and a matter correlator. The expression for the matter correlator is fixed by the conformal invariance of the worldsheet and boundary theories as follows:

$$\begin{aligned} \Big\langle \mathcal{V}_1(x_1;z_1)\mathcal{V}_3(x_3;z_3)\mathcal{V}_4^{\mu_1\cdots\mu_l}(x_4;z_4)\Big\rangle &= \frac{\bar{F}_{134}^{(0,0,l)}}{|z_{13}|^{h_1+h_3-h_4}|z_{34}|^{h_3+h_4-h_1}|z_{14}|^{h_1+h_4-h_3}} , \\ &\quad \frac{\Big(V^{\mu_1}(x_1,x_3,x_4)\cdots V^{\mu_l}(x_1,x_3,x_4) - \text{Traces}\Big)}{|x_{13}|^{\Delta_1+\Delta_3-\Delta_4+l}|x_{34}|^{\Delta_3+\Delta_4-\Delta_1-l}|x_{14}|^{\Delta_4+\Delta_1-\Delta_3-l}} . \end{aligned} \tag{5.46}$$

The ghost correlator is given by

$$\langle \bar{c}c(z_1)\bar{c}c(z_3)\bar{c}c(z_4)\rangle = C^g_{S_2}|z_{13}|^2|z_{34}|^2|z_{14}|^2\,, \tag{5.47}$$

where $C^g_{S_2}$ is the ghost normalization factor.

We also want the relation between three-point function coefficients $\bar{F}^{(0,0,l)}_{34\Delta_0}$ and OPE coefficients $F^{(0,0,l)}_{34\Delta_0}$. This is derived in appendix. H. We state the relation here:

$$\bar{F}^{(0,0,l)}_{34\Delta_0} = \left[\pi^{d/2}\Lambda_l(\beta',\gamma')D_l(\Delta_0)\frac{\Gamma\left(\frac{d-\beta'}{2}\right)\Gamma\left(\frac{d-\gamma'}{2}\right)\Gamma\left(\frac{\beta'+\gamma'-d}{2}+l\right)}{\Gamma\left(\frac{\beta'}{2}+l\right)\Gamma\left(\frac{\gamma'}{2}+l\right)\Gamma\left(\frac{2d-\beta'-\gamma'}{2}\right)}\right]F^{(0,0,l)}_{34\Delta_0}\,, \tag{5.48}$$

where $\beta' = \Delta_4 - \Delta_3 - \Delta_0 + d - l$ and $\gamma' = \Delta_3 - \Delta_4 - \Delta_0 + d - l$ as given in eq. (5.37), and $\Lambda_l(a,b)$ is defined in eq. (5.35). $D_l(\Delta_i)$ is a constant depending upon the conformal dimension that appears in the two-point function of vertex operators of dimension Δ_i as follows:

$$\left\langle \mathcal{V}_1^{\mu_1\cdots\mu_l}(x_1,z_1)\mathcal{V}_{2,\nu_1\cdots\nu_l}(x_2,z_2)\right\rangle = \frac{\delta_{q_1,q_2}}{|z_{12}|^{2h_1}}\left[D_l(\Delta_1)\delta(\Delta_1-\Delta_2)\frac{J^{(\mu_1}_{\nu_1}\cdots J^{\mu_l)}_{\nu_l} - \text{Traces}}{|x_{12}|^{2\Delta_1}}\right]\,, \tag{5.49}$$

where q_1 and q_2 are additional discrete quantum numbers (suppressed on the left hand side of eq. (H.2)) that maybe associated with the vertex operators $\mathcal{V}_1^{\mu_1\cdots\mu_l}$ and $\mathcal{V}_{2,\nu_1\cdots\nu_l}$ respectively.

Now we have the ingredients to calculate the three-point function in eq. (5.45). We fix the points z_1, z_3 and z_4 at $0, 1$ and ∞ respectively and use eq. (H.4) for the matter correlator, eq. (5.47) for the ghosts, and the relation between the three-point function coefficient $\bar{F}^{(0,0,l)}_{34\Delta_0}$ and the OPE coefficient $F^{(0,0,l)}_{34\Delta_0}$ as given in eq. (5.48), we obtain:

$$\begin{aligned}
&\left\langle \bar{c}c\mathcal{V}^{\mu_1\cdots\mu_l}_{\Delta_0,q}(0;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\right\rangle \\
&= C^g_{S_2}\bar{F}^{(0,0,l)}_{34\Delta_0}\lim_{z_4\to\infty}|z_4|^{-2(h_4-2)}\frac{V^{\mu_1}(x_3,x_4,0)\cdots V^{\mu_l}(x_3,x_4,0) - \text{Traces}}{|x_3|^{(\Delta_3+\Delta_0-\Delta_4)-l}|x_4|^{(\Delta_4+\Delta_0-\Delta_3)-l}|x_3-x_4|^{(\Delta_3+\Delta_4-\Delta_0)+l}}\,, \\
&= F^{(0,0,l)}_{34\Delta_0}C^g_{S_2}\left[\pi^{d/2}\Lambda_l(\beta',\gamma')D_l(\Delta_0)\frac{\Gamma\left(\frac{d-\beta'}{2}\right)\Gamma\left(\frac{d-\gamma'}{2}\right)\Gamma\left(\frac{\beta'+\gamma'-d}{2}+l\right)}{\Gamma\left(\frac{\beta'}{2}+l\right)\Gamma\left(\frac{\gamma'}{2}+l\right)\Gamma\left(\frac{2d-\beta'-\gamma'}{2}\right)}\right] \\
&\qquad \frac{V^{\mu_1}(x_3,x_4,0)\cdots V^{\mu_l}(x_3,x_4,0) - \text{Traces}}{|x_3|^{(\Delta_3+\Delta_0-\Delta_4)-l}|x_4|^{(\Delta_4+\Delta_0-\Delta_3)-l}|x_3-x_4|^{(\Delta_3+\Delta_4-\Delta_0)+l}}\,.
\end{aligned} \tag{5.50}$$

Using the expression in eq. (5.50) to obtain $\bar{B}$ in eq. (5.45) and then comparing eq. (5.43) and eq. (5.44), we finally obtain the following explicit relationship between the OPE coefficient $\lambda^{(0,0,l)}_{12\Delta_0}$ of the boundary CFT and the corresponding worldsheet OPE coefficient $F^{(0,0,l)}_{12\Delta_0}$:

$$\lambda^{(0,0,l)}_{12\Delta_0} = \left(\sqrt{\frac{-4i\pi^{d+2}C^g_{S_2}D_l(\Delta_0)}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}K_l(\Delta_0)}}\;\frac{I_{12}}{a_1^{(0)}a_2^{(0)}}\right)F^{(0,0,l)}_{12\Delta_0}\,. \tag{5.51}$$

Let us recall that in the above expression $C^g_{S_2}$ is the normalization of the ghost correlator defined through eq. (5.47), $D_l(\Delta_0)$ is the coefficient in the two-point function of the worldsheet operators $\mathcal{V}_{\Delta_0,q}$. $K_l(\Delta_0)$ appears as coefficient of the two-point function of boundary spin l operators as defined in eq. (5.41). The normalization factors $a_i^{(0)}$ are specified in eq. (I.3) and I_{12} is given by,

$$\begin{aligned} I_{12} &= \frac{\Gamma\left(\frac{\Delta_0+\Delta_1-\Delta_2+l}{2}\right)\Gamma\left(\frac{\Delta_0+\Delta_2-\Delta_1+l}{2}\right)\Gamma\left(\frac{d-2\Delta_0}{2}\right)}{\Gamma\left(\frac{\Delta_1-\Delta_2-\Delta_0+d+l}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1-\Delta_0+d+l}{2}\right)\Gamma\left(l+\Delta_0\right)} \\ &\quad \Lambda_l\left(\Delta_2-\Delta_1-\Delta_0+d-l,\Delta_1-\Delta_2-\Delta_0+d-l\right). \end{aligned} \tag{5.52}$$

5.4 Generalization to spinning correlators

We can repeat the analysis of the previous section for correlation functions of spinning operators. For simplicity, let us consider the four-point function of two conserved vector currents and two scalar operators and consider the exchange of a scalar in the OPE of the two vectors. Our goal will again be to obtain the relationship between the vector-vector-scalar OPE coefficients in the worldsheet and the boundary theories.

Let us first look at the OPE of two conserved vectors in the boundary theory. A conserved current $\mathcal{O}_\Delta^{\mu_1\cdots\mu_l}$ has dimension $\Delta = l+d-2$. For global symmetry currents ($l=1, \Delta_1 = \Delta_2 = d-1$), we have the following result [162] for the three-point function of two currents with a scalar,

$$\langle \mathcal{O}^\mu(x_1)\mathcal{O}^\nu(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \bar{\lambda}_{123}^{(1,1,0)}\frac{J^\mu{}_\sigma(x_1-x_3)J^{\sigma\nu}(x_3-x_2)+c\ J^{\mu\nu}(x_1-x_2)}{|x_{12}|^{2(d-1)-\Delta_3}|x_{23}|^{\Delta_3}|x_{13}|^{\Delta_3}}\,, \tag{5.53}$$

where $J^{\mu\nu}$ is given by eq. (5.13) and the constant c is given by

$$c = \frac{\Delta_3-2(d-1)}{\Delta_3}\,. \tag{5.54}$$

From eq. (5.53), we can obtain the contribution of a scalar operator to the OPE of two conserved currents as follows,

$$\mathcal{O}_1^\mu(x_1)\mathcal{O}_2^\nu(0) \supset \sum_\Delta \lambda_{12\Delta}^{(1,1,0)}\ \frac{\hat{x}_1^\mu\hat{x}_1^\nu + b\ \delta^{\mu\nu}}{|x_1|^{\Delta_1+\Delta_2-\Delta}}\mathcal{O}_\Delta(0)+\cdots\,, \tag{5.55}$$

with,

$$b = -\frac{(\Delta+1-d)}{\Delta+2-2d} \quad \text{and} \quad \bar{\lambda}_{123}^{(1,1,0)} = \lambda_{123}^{(1,1,0)}\,. \tag{5.56}$$

Therefore the contribution of a scalar exchange to the four-point function of two such operators and two scalar operators with scaling dimensions Δ_3 and Δ_4 can be evaluated as,

$$\begin{aligned} &\langle \mathcal{O}^\mu(x_1)\mathcal{O}^\nu(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \\ &\supset \sum_\Delta \lambda_{12\Delta}^{(1,1,0)}\frac{b\ \delta^{\mu\nu}+\hat{x}_1^\mu\hat{x}_1^\nu}{|x_1|^{2(d-1)-\Delta}}\langle \mathcal{O}_\Delta(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle\,, \\ &= \sum_\Delta \frac{b\ \delta^{\mu\nu}+\hat{x}_1^\mu\hat{x}_1^\nu}{|x_1|^{2(d-1)-\Delta}}\frac{\lambda_{12\Delta}^{(1,1,0)}\lambda_{34\Delta}^{(0,0,0)}}{|x_3|^{\Delta_3+\Delta-\Delta_4}|x_4|^{\Delta_4+\Delta-\Delta_3}|x_{34}|^{\Delta_3+\Delta_4-\Delta}}\,. \end{aligned} \tag{5.57}$$

Again, we can represent the boundary operators as the integrated worldsheet vertex operators that carry the boundary Lorentz indices as global symmetry labels and approximate the four-point function using the worldsheet OPE analysis. Let us consider the worldsheet OPE of such vertex operators and focus on the contribution of vertex operators dual to scalars in the boundary CFT. The most general form of this term (ignoring the contribution from the descendants) can be written as,

$$\mathcal{V}_1^\mu(x_1,z_1)\mathcal{V}_2^\nu(x_2,0)\supset\sum_q\int_C d\Delta\int d^dx\; F^{\mu\nu}(z_1;x_i,x;\Delta_i,\Delta;q_i,q)\mathcal{V}_{\Delta,q}(x,0)\,. \tag{5.58}$$

The x and z dependence of $F^{\mu\nu}$ is fixed by the conformal invariance and we can write the OPE as follows:

$$\begin{aligned}&\mathcal{V}_1^\mu(x_1,z_1)\mathcal{V}_2^\nu(x_2,0)\\ \supset\;&\sum_q\int_C d\Delta\int d^dx\,\frac{F_{12\Delta}^{(1,1,0)}}{|z_1|^{h_1+h_2-h_{\Delta,q}}}\frac{\left(\bar{c}J^{\mu\nu}(x_1-x_2)+J^\mu{}_\sigma(x_1-x)J^{\sigma\nu}(x_2-x)\right)}{|x_1-x_2|^\alpha|x_2-x|^\beta|x_1-x|^\gamma}\mathcal{V}_{\Delta,q}(x,0)\,,\end{aligned} \tag{5.59}$$

with,

$$\alpha=2(d-1)+\Delta-d\,,\qquad\qquad\beta=\gamma=d-\Delta\,. \tag{5.60}$$

When $\mathcal{V}_i^\mu$ satisfy the current conservation equation on the boundary, $\bar{c}$ is fixed and can be obtained from eq. (5.54) with Δ replaced by $d-\Delta$ to give,

$$\bar{c}=\frac{2-d-\Delta}{d-\Delta}\,. \tag{5.61}$$

We shall now set $x_2=0$ and further approximate the contribution to the OPE in eq. (5.59) by changing variables to $y=\frac{x}{|x_1|}$ and keeping only the leading order piece around $x_1=0$. This gives us, after evaluating the x integral,

$$\mathcal{V}_1^\mu(x_1,z_1)\mathcal{V}_2^\nu(0,0)\;\supset\;\sum_q\int_C d\Delta\frac{|z_1|^{-(h(1)+h(2)-h(\Delta,q))}}{|x_1|^{\alpha+\beta+\gamma-d}}F_{12\Delta}^{(1,1,0)}\;\mathcal{V}_{\Delta,q}(0,0)\;H^{\mu\nu}(x_1)\,, \tag{5.62}$$

with $H^{\mu\nu}$ being given by the integrals in appendix. J as,

$$H^{\mu\nu}(x)\;=\;\frac{\pi^{d/2}\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma\left(\frac{d-\Delta}{2}\right)\Gamma\left(\frac{d-\Delta}{2}\right)\Gamma\left(\Delta\right)}\left[\frac{-2\Delta(\Delta+2-2d)}{(d-\Delta)^2(\Delta+1-d)}\right]\left[\hat{x}^\mu\hat{x}^\nu-\frac{(\Delta+1-d)}{(\Delta+2-2d)}\delta^{\mu\nu}\right]. \tag{5.63}$$

Note that the ratio of the coefficients of $\delta^{\mu\nu}$ and $\hat{x}_1^\mu\hat{x}_1^\nu$ in eq. (5.63) is precisely the corresponding ratio in eq. (5.56) in the boundary OPE in eq. (5.55).

Now we can use the normalized expression for the boundary CFT operators as integrated vertex operators as given in eq. (I.2) and approximate it with the OPE in eq. (5.62) and the saddle point analysis discussed in sec. 5.1. This gives us:

$$\begin{aligned}
&\big\langle \mathcal{O}^\mu(x_1)\mathcal{O}^\nu(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\big\rangle \\
&= \frac{1}{a_1^{(1)}a_2^{(1)}a_3^{(0)}a_4^{(0)}}\int d^2z \langle \mathcal{V}^\mu(x_1;z)\bar{c}c\mathcal{V}^\nu(0;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\rangle\,, \\
&\supset -\frac{2i\pi}{a_1^{(1)}a_2^{(1)}a_3^{(0)}a_4^{(0)}}\sum_q \frac{|x_1|^{\Delta-2(d-1)}}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}} B^{\mu\nu}(\Delta_i,q_i;\Delta_0,q)\,,
\end{aligned} \tag{5.64}$$

where, the function $B^{\mu\nu}$ is given by,

$$B^{\mu\nu} = 2\pi F^{(1,1,0)}_{12\Delta_0}\,\langle \bar{c}c\mathcal{V}_{\Delta,q}(0;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\rangle\, H^{\mu\nu}(x_1)\,. \tag{5.65}$$

and $H^{\mu\nu}$ is given by eq. (5.63) with Δ set to Δ_0.

The three-point function appearing in eq. (5.65) can be obtained from eq. (5.50) with $l=0$. Thereafter, we can compare eq. (5.64) with the corresponding result from the boundary OPE analysis in eq. (5.57). This gives us the following relation between the vector-vector-scalar OPE coefficients in the boundary and worldsheet theories,

$$\lambda^{(1,1,0)}_{12\Delta_0} = \left(\sqrt{\frac{-4i\pi^{d+2}C^g_{S_2}D_0(\Delta_0)}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}}}\,\frac{\bar{I}_{12}}{a_1^{(1)}a_2^{(1)}}\right)F^{(1,1,0)}_{12\Delta_0}\,, \tag{5.66}$$

where,

$$\bar{I}_{12} = \left[\frac{2\Delta_0(2d-\Delta_0-2)}{(d-\Delta_0)^2(\Delta_0+1-d)}\right]\frac{\Gamma\left(\frac{\Delta_0}{2}\right)\Gamma\left(\frac{\Delta_0}{2}\right)\Gamma\left(\frac{d-2\Delta_0}{2}\right)}{\Gamma\left(\frac{d-\Delta_0}{2}\right)\Gamma\left(\frac{d-\Delta_0}{2}\right)\Gamma\left(\Delta_0\right)}\,. \tag{5.67}$$

We can also derive the relation between the all scalar OPE coefficients in the boundary and worldsheet theories from the spinning correlator considered in this section and the result is consistent with eq. (5.51).

5.5 Coupling Constants in AdS supergravity

In this section, we quote some relations between cubic couplings in AdS supergravity and the OPE coefficients in the boundary CFT. These relations, for the coupling $g^{(0,0,l)}_{\phi\phi h}$ between two scalars ϕ_1, ϕ_2 and one spin l field $h^{\mu_1\cdots\mu_l}$ in AdS supergravity, were obtained in [144]. The interaction term is taken to be of the form $\frac{1}{S_{\phi\phi h}}g^{(0,0,l)}_{\phi\phi h}\phi_1\left(\nabla_{\mu_1\cdots\mu_l}\phi_2\right)h^{\mu_1\cdots\mu_l}$, $S_{\phi\phi h}$ being a symmetry factor. We quote the result here,

$$\begin{aligned}
\lambda^{(0,0,l)}_{123} = g^{(0,0,l)}_{\phi\phi h}\frac{\pi^{\frac{d}{2}}\sqrt{C_{\Delta_1}C_{\Delta_2}C_{\Delta_3,l}}}{2^{1-l}\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3+l)}\Gamma\left(\alpha_{12}+\frac{l}{2}\right)\Gamma\left(\alpha_{23}+\frac{l}{2}\right)\Gamma\left(\alpha_{13}+\frac{l}{2}\right) \\
\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3+l-d}{2}\right)\,,
\end{aligned} \tag{5.68}$$

where, Δ_1, Δ_2 and Δ_3 are the scaling dimensions of the boundary operators dual to the AdS fields ϕ_1, ϕ_2 and $h^{\mu_1 \cdots \mu_l}$ respectively and

$$\alpha_{12} = \frac{\Delta_1 + \Delta_2 - \Delta_3}{2} \quad , \qquad \alpha_{23} = \frac{\Delta_2 + \Delta_3 - \Delta_1}{2} \quad , \quad \alpha_{13} = \frac{\Delta_1 + \Delta_3 - \Delta_2}{2} \, ,$$

$$C_{\Delta,l} \;=\; \frac{1}{2}\left(\frac{l+\Delta-1}{\Delta-1}\right)\frac{\pi^{-d/2}\Gamma(\Delta)}{\Gamma\left(\Delta+1-\frac{d}{2}\right)} \, , \quad C_{\Delta,0} \equiv C_\Delta \tag{5.69}$$

We shall also state the result for a Witten diagram with two external conserved currents (of dimension $d-1$)and a scalar. The interaction term (see [136]) considered is of the form $\eta^{\mu\nu}\eta^{\rho\sigma}\phi\partial_{[\mu}A_{\nu]}\partial_{[\rho}A_{\sigma]}$. This diagram can be computed by an inversion on the coordinates (which is an isometry of AdS) followed by a direct evaluation of the integrals. The relation between the corresponding OPE coefficient and the cubic coupling in supergravity is given by,

$$\lambda_{123}^{(1,1,0)} = g_{123}^{(1,1,0)} \pi^{\frac{d}{2}} \sqrt{C_{d-1,1}C_{d-1,1}C_\Delta} \frac{(d-2)^2 \Gamma\left(\frac{\Delta}{2}+1\right)\Gamma\left(\frac{\Delta+d-2}{2}\right)\Gamma\left(\frac{2d-2-\Delta}{2}\right)}{2\Gamma(\Delta)\Gamma(d)\Gamma(d)} \, . \tag{5.70}$$

The results presented here all assume a canonical normalization for terms in the Lagrangian as mentioned before and the overall factors in the Witten diagrams are adjusted to ensure consistency with a unit normalized two-point function. In general, we can have cubic vertices different from the ones considered here in which case the relation between the bulk cubic coupling and the OPE coefficient gets modified. Such results involving one or more higher spin gauge fields in the bulk can be obtained from work presented in [101, 221–225].

We should also mention that the relations between the bulk cubic couplings and the boundary OPE coefficients are not applicable for extremal correlators since the bulk extremal cubic couplings vanish. One needs to use appropriate boundary terms to obtain the corresponding boundary OPE coefficients (see, e.g., [125, 226]).

The results from sec. 5.3, sec. 5.4 and the current section thus give a three way relation between cubic couplings in the bulk supergravity, OPE coefficients in the boundary CFT and OPE coefficients in the worldsheet CFT for operators whose two and three-point functions enjoy a non-renormalization theorem.

Chapter 6

Conclusion

In this thesis, we explored some analytical approaches to study CFTs and the associated OPE data. OPE data constituting of dimensions of operators and OPE coefficients defines a CFT non-perturbatively and as such are of focal interest in the subject. Owing to the presence of conformal symmetry, a lot can be deduced about CFTs in a completely theory independent manner as direct or indirect consequences of symmetry alone. Indeed, this has largely been the guiding principle behind the results presented here.

To begin with, we proved some universal features about the spectrum of local operators supported on a conformal defect based on the convergent OPE and its associativity. The central object of our analysis here was the two-point function of scalar operators in the ambient Lorentzian theory. We showed that satisfying crossing symmetry of this correlator with almost lightlike separated operators implies the existence of a countably infinite number of universal accumulation points in the transverse twist $\widehat{\Delta} - s$ spectrum of the defect theory, each accumulation point being populated by infinite towers of transverse derivative operators with asymptotically large s. Finite s corrections to the universal limiting values of operator dimensions and OPE coefficients can be obtained by solving the crossing equation in a double lightcone expansion. Although these lightcone bootstrap techniques are based on a few assumptions, the results derived therefrom are put on a more rigorous footing by the Lorentzian inverse to the defect channel expansion of the bulk two-point function. This inversion formula extracts the OPE data in the two-point function from an integral over the discontinuity in the causal correlator. The integral kernel is analytic in s and this establishes the phenomenon of analyticity in s of the defect channel OPE data. We applied the inversion formula considering the two-point function in the bulk OPE limit thereby resumming the results from the lightcone bootstrap. Finally, we demonstrated these results in the context of a free theory with a defect and the twist defect in the three dimensional Ising CFT.

The Lorentzian formula holds true over a certain lower bound for s, say s^*. The primary shortcoming in our work on the inversion formula lies in the fact that we were not able to derive a universal upper bound on s^*. If the two-point function in a theory is not polynomially bounded in a certain kinematic region, this theory may not be featuring the towers of transverse derivative operators in its spectrum. It would be interesting to put this aspect of the inversion formula on stronger ground by either deriving a model independent upper bound on s^* or showing with an example that such a bound need not exist. To further elucidate this matter, we would need to understand the corrections to the inversion formula for $s < s^*$. While applying the inversion formula in the bulk OPE limit, we also worked in an expansion around $z = 0$ although this expansion does not commute with the block expansion. We justified this by choosing to work in the regime of small corrections such as the large s regime. The OPE inversion formula is however generally applicable and as such it is important to account for the error from commuting the small z expansion with the bulk OPE expansion - see [74] for the relevant procedure in the context of the four-point function in CFTs without defects.

Following our discussion on defect CFTs and the associated CFT data, we turned our attention to Mellin amplitudes. Mellin amplitudes encode data on operator dimensions and OPE coefficients in their poles and residues and have factorization properties analogous to those of scattering amplitudes in massive QFT. We introduced Mellin amplitudes for fermionic correlation functions as the data corresponding to the fermionic sector of a CFT is not encoded in Mellin amplitudes of bosonic operators. These Mellin amplitudes have multiple components, each associated with an independent tensor structure. We argued that not all choices of bases (of tensor structures) is suitable to define Mellin amplitudes and addressed the relevant subtleties. For four-point functions of scalars and spin half fermions in three dimensions, we worked out the pole structure of the Mellin amplitudes while also making their factorization properties manifest. Thereafter, we explicitly computed the Mellin amplitudes for a few Witten diagrams and conformal Feynman integrals with fermionic legs, thereby demonstrating the general properties of fermionic Mellin amplitudes.

The Mellin amplitude for a four-point function of scalars has an infinite series of poles corresponding to each conformal family contributing to the correlator via the OPE. From the leading pole of a given series, the twist of the corresponding primary operator can be read off while the degree of the polynomial residue tells us the spin, thus giving us the dimension of the primary. The OPE coefficient corresponding to this conformal family is encoded in the residue corresponding to this pole and can be read off given that we know the kinematically fixed form of the polynomial in the residue. The pole structure of a fermionic Mellin amplitude is more involved as each component may now have more than one series of poles for each primary in the OPE. For example, the Mellin amplitudes we have discussed generically have two series of poles in each of their components for a given primary contributing to the OPE. If the relevant three-point functions are of definite parity, then we have simplifications as one of the series of poles is absent. In general, the twists of the exchanged primaries are given by the leading poles and the residues from the different series of poles give us different OPE coefficients. It is however important to determine the functional form of the polynomials that appear as residues and this should complete our toolbox to study fermionic CFTs through the Mellin representation. We should also emphasize on the fact that the exact pole structure (and analytic properties in general) of these Mellin amplitudes depends on the choice of basis (of tensor structures). It may be possible to choose a basis such that the tensor structures in the four-point functions and the contributing conformal blocks have a perfect alignment such that we have only a single series of poles per primary in each of the components of the Mellin amplitude (at least in a given channel). It would be interesting to carry out a careful analysis of this possibility.

Finally, we had a discussion on the OPE in the light of the AdS/CFT correspondence and its implications for OPE coefficients in the worldsheet CFT of a string theory in AdS and the dual boundary CFT. It was shown in [145] that the contribution of a scalar primary to the OPE of two scalars in the boundary theory can be reproduced from the OPE of the dual vertex operators in the worldsheet CFT in a completely theory independent manner. We generalized this work to explain all contributions (from integer spin primaries) to the OPE of two scalars in the boundary theory from a worldsheet perspective. We also showed that the same principles apply to the OPE of spinning operators as we reproduced the contribution of a scalar primary to the OPE of two conserved spin one currents in the boundary CFT from the OPE of dual vertex operators in the worldsheet CFT. Furthermore, we used this analysis to obtain a set of relations between OPE coefficients in the worldsheet CFT of the AdS string theory and OPE coefficients in the dual boundary CFT. It may happen that certain sectors of the CFT data for the theory living on the conformal boundary of AdS enjoys some non-renormalization theorems owing to the presence of a high degree of symmetry. For example, three-point functions of chiral primaries in four dimensional $\mathcal{N} = 4$ SYM at large N are not renormalized [146]. In such cases, our OPE relations and relations between cubic couplings in bulk supergravity and OPE coefficients in the boundary CFT (see for example [144]) relate data from different regimes of the duality.

So far we have only considered the contribution of single-trace operators to the OPE in the boundary CFT. Multi-trace operators contribute to the OPE as well and these typically involve logarithmic terms. In the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$, it was shown in [121] that the contribution of multi-trace operators to the OPE in the boundary CFT can be associated with discrete contributions generated during the analytic continuation (see sec. 5.1) of the OPE of the normalizable vertex operators in the worldsheet CFT. It was however noted in [145] that such terms cannot always be interpreted as the contribution of multi-trace operators. It is possible that the contributions of multi-trace operators manifest themselves in the worldsheet CFT in a non-local manner and the treatment based on the local OPE in our work here may not be adequate to capture them. It would be interesting to investigate into this matter in order to have a complete understanding of how the OPE in the boundary CFT emerges from physics on the worldsheet in the dual AdS string theory.

The results discussed in this thesis point at several future directions of research. It would be interesting to apply lightcone bootstrap and the inversion formula to study the large s spectrum of strongly interacting defect theories or go beyond the leading order in perturbation theory. Since the inversion formula in the case of defect CFTs is obtained in a simpler manner as compared to the inversion formula associated with four-point functions in CFTs without defects, this setting offers us the opportunity to understand the origins and the implications of the OPE inversion formula better. It can be expected that the spectrum of local operators supported on the defect has accumulation points dictated by the spinning operators in the bulk CFT as well. It would be interesting to generalize the inversion formula that we presented to the case of two-point functions of spinning bulk operators and show the existence of these accumulation points rigorously. One more important problem suggested by our work on defect CFTs is to derive an inversion formula for the bulk channel expansion of the two-point function of bulk operators. Note that in the case of the four-point function in CFTs without defects, all the channels are essentially the same and by applying the inversion formula in two different channels in a loop (in the spirit of [33]) we can approximately solve the crossing equation using data on a few operators. To set up an analogous procedure for defect CFTs, it is esstential to have the inversion formula for the bulk channel too and apply it to extract bulk channel data from the correlator in the defect OPE limit. The bulk channel inversion formula should of course capture the same spectrum as the inversion formula for the four-point function in theories without defects, although the associated OPE coefficients are now different. In fact, a rudimentary lightcone bootstrap analysis shows that contribution of the the double twist operators to the bulk channel expansion is sufficient to satisfy the crossing equation in this lightcone limit with a bulk operator almost lightlike separated from the defect.

Furthermore, it would be interesting to push the recent extension of the Mellin formalism to defect CFTs [110, 111] further and employ Mellin bootstrap techniques for defect CFTs using $\frac{1}{s}$ as a small parameter, similar to the application of the large spin expansion to Mellin bootstrap in CFTs without defects [107]. Our work of Mellin amplitudes for fermionic correlation functions needs to be complemented with a rigorous derivation of the polynomial residues of the Mellin amplitudes and this would set the stage for the application of Mellin bootstrap to fermionic CFTs such as the Gross-Neveu model in three dimensions or the Gross-Neveu-Yukawa theory in $4 - \epsilon$ dimensions. It would also be interesting to pursue an independent study of Witten diagrams (with or without loops) with fermionic legs and in particular, look at the flat space limit of these Mellin amplitudes. An important aspect of Mellin amplitudes for spinning correlators, that perhaps leaves it a bit usatisfying, is the fact there seems to be no canonical definition of such Mellin amplitudes. Spinning correlators are expanded in a basis of tensor structures and the choice of a basis is central to defining the associated Mellin amplitudes. The Mellin amplitude which is now a set of functions is also not amenable to a direct interpretation as being part of a vector space. Even a simple linear combination of tensor structures with the coefficients given by products of cross-ratios results in a linear combination of different components of the Mellin amplitude *and* a shift in their arguments. It is therefore important to investigate whether Mellin amplitudes can be defined for spinning correlators without an a-priori prescription for the basis of tensor structures. This would also pave the way to incorporate operators of any value of spin into the Mellin formalism.

Our efforts to study defect CFTs through the conformal bootstrap, develop the Mellin formalism for fermionic correlators and understand the OPE in a holographic context are all part of the broader goal of exploring the ramifications of conformal symmetry in QFT and its role in holography. How far can we proceed in constraining the space of CFTs (and thus the space of QFTs) based on the consequences of symmetries alone? Can we relate the consistency conditions in a worldsheet CFT of an AdS string theory to those in the boundary CFT thus quantifying the importance of conformal symmetry [227] in holographic dualities? The jury is still out on these questions and we are sure to have interesting days ahead.

Appendix A

Defect channel conformal blocks

As discussed in sec. 2.5.1, the two-point function of bulk scalars $\langle\phi(x_1)\,\phi(x_2)\rangle$ in a CFT with a defect can be expanded in blocks using the defect channel OPE in eq. (2.74). The corresponding defect channel conformal blocks are eigenfunctions of the quadratic Casimir of $SO(p+1,1)\times SO(q)$ where p and q are the dimension and the codimension of the flat defect in the theory. These eigenfunctions can be derived to be of the following form as given in eq. (2.91),

$$\widehat{g}_{\widehat{\Delta},s}(\chi,\theta)=\chi^{-\widehat{\Delta}}\,{}_2F_1\left(\frac{\widehat{\Delta}}{2}+\frac{1}{2},\frac{\widehat{\Delta}}{2},\widehat{\Delta}+1-\frac{p}{2},\frac{4}{\chi^2}\right)\begin{pmatrix}s+\frac{q}{2}-2\\ \frac{q}{2}-2\end{pmatrix}^{-1}C_s^{\left(\frac{q}{2}-1\right)}(\cos\theta)\ , \tag{A.1}$$

where the variables χ,θ are invariants of the symmetry group mentioned above, and are given as follows,

$$\begin{aligned}\chi\quad &=\quad \frac{x_{12}^2-2\,(x_{12})_i\,(x_{12})^i}{|(x_1)_\perp||(x_2)_\perp|}\,,\\ \eta\quad &=\cos\theta=\quad\frac{(x_{12})_i\,(x_{12})^i}{|(x_1)_\perp||(x_2)_\perp|}\,.\end{aligned} \tag{A.2}$$

x_{12}^2 is the distance square betwen x_1 and x_2. Indices i,j label directions orthogonal to the defect while indices a,b label the ones parallel to the defect. $|x_\perp|$ is the perpendicular distance of the bulk point x from the defect.

Let us now see how we can go from the block in eq. (A.1) to the expression presented in eq. (2.93). We want to express the radial part of the block in the variable r with,

$$\chi=r+\frac{1}{r}\,. \tag{A.3}$$

Let us note the following quadratic transformation of the hypergeometric function [228],

$$(1+z)^{-a}\,{}_2F_1\left(\frac{a+1}{2},\frac{a}{2},a-b+1,\frac{4z}{(1+z)^2}\right)=\,{}_2F_1\left(a,b,a-b+1,z\right)\,. \tag{A.4}$$

Using $a=\widehat{\Delta}$, $b=p$ and $z=r^2$ in eq. (A.4) gives us,

$$\chi^{-\widehat{\Delta}}\,{}_2F_1\left(\frac{\widehat{\Delta}}{2}+\frac{1}{2},\frac{\widehat{\Delta}}{2},\widehat{\Delta}+1-\frac{p}{2},\frac{4}{\chi^2}\right)=r^{\widehat{\Delta}}\,{}_2F_1\left(\widehat{\Delta},\frac{p}{2},\widehat{\Delta}+1-\frac{p}{2},r^2\right)\,. \tag{A.5}$$

Now we turn to the angular part of the block in eq. (A.1). We can express the Gegenbauer polynomial as a hypergeometric function with the following identity,

$$C_n^{(\alpha)}(1-2x) = \frac{(2\alpha)_n}{n!}\,{}_2F_1\left(-n, n+2\alpha, \alpha+\frac{1}{2}, x\right) , \tag{A.6}$$

where n is a positive integer. With $n = s$, $\alpha = \frac{q}{2} - 1$ and $x = \frac{1-\eta}{2}$, we get,

$$\binom{s+\frac{q}{2}-2}{\frac{q}{2}-2}^{-1} C_s^{\left(\frac{q}{2}-1\right)}(\eta) = \frac{\Gamma\left(\frac{q}{2}-1\right)\Gamma\left(s+q-2\right)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(q-2\right)}\,{}_2F_1\left(-s, s+q-2, \frac{q-1}{2}, \frac{1-\eta}{2}\right) . \tag{A.7}$$

We wish to change to variable w such that,

$$\eta = \frac{1}{2}\left(w + \frac{1}{w}\right) . \tag{A.8}$$

After this variable change, let us now use the following quadratic transformation of the hypergeometric function [228],

$$(1-z)^{-\frac{a}{2}}\,{}_2F_1\left(a, 2b-a, b+\frac{1}{2}, -\frac{\left(1-\sqrt{1-z}\right)^2}{4\sqrt{1-z}}\right) = {}_2F_1\left(a, b, 2b, z\right) . \tag{A.9}$$

Using eq. (A.9) with $a = -s$, $b = \frac{q}{2} - 1$ and $z = 1 - w^2$, we get,

$${}_2F_1\left(-s, s+q-2, \frac{q-1}{2}, \frac{1-\eta}{2}\right) = w^{-s}\,{}_2F_1\left(-s, \frac{q}{2}-1, q-2, 1-w^2\right) . \tag{A.10}$$

Next we want to use the following linear hypergeometric transformation [228],

$$\begin{aligned} {}_2F_1\left(a, b, c, z\right) \;=\; & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a, b, a+b-c+1, 1-z) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c-a, c-b, c-a-b+1, 1-z)\,, |\mathrm{Arg}(1-z)| < \pi \end{aligned} \tag{A.11}$$

with $a = -s$, $b = \frac{q}{2} - 1$, $c = q - 2$ and $z = 1 - w^2$. Note that if s is an integer and q is not even, the second term on the right hand side of eq. (A.11) goes to zero. Assuming this, we can apply eq. (A.11) on the the right hand side of eq. (A.10), to finally obtain,

$$\binom{s+\frac{q}{2}-2}{\frac{q}{2}-2}^{-1} C_s^{\left(\frac{q}{2}-1\right)}(\eta) = w^{-s}\,{}_2F_1\left(-s, \frac{q}{2}-1, 2-\frac{q}{2}-s, w^2\right) . \tag{A.12}$$

Eq. (A.5) and eq. (A.12) give us the representation of the block in eq. (2.93).

$$\widehat{g}_{\widehat{\Delta},s}(r, w) = r^{\widehat{\Delta}} w^{-s}\,{}_2F_1\left(-s, \frac{q}{2}-1, 2-\frac{q}{2}-s, w^2\right)\,{}_2F_1\left(\widehat{\Delta}, \frac{p}{2}, \widehat{\Delta}+1-\frac{p}{2}, r^2\right) . \tag{A.13}$$

Note that although we assumed that s is an integer and q is not an even integer to obtain eq. (A.13) from eq. (A.1), now we consider the block in eq. (A.13) to be defined for all s and all q. In the configuration described in sec. 3.2, $r^2 = z\bar{z}$ and $w^2 = \frac{z}{\bar{z}}$ and using this in eq. (A.13) gives us the blocks in eq. (3.8).

Appendix B

Hypergeometric identities

We wish to prove the hypergeometric identities in eq. (3.66) and eq. (3.74) that were used in the derivation of the Lorentzian inverse to the defect channel OPE expansion of the bulk two-point function in sec. 3.4.2. We start with the following identity,

$$\begin{aligned} {}_2F_1\left(a,b;c;z\right) &= \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}\,{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{z}\right) \\ &\quad+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}\,{}_2F_1\left(b-c+1,b;-a+b+1;\frac{1}{z}\right), \qquad z\notin(0,1)\,. \end{aligned} \tag{B.1}$$

Now we put in eq. (B.1),

$$a=s+q-2\,,\qquad b=\frac{q}{2}-1\,,\qquad c=\frac{q}{2}+s\,,\qquad z=\frac{1}{w^2}\,, \tag{B.2}$$

which gives,

$$\begin{aligned} &{}_2F_1\left(s+q-2,\frac{q}{2}-1;\frac{q}{2}+s;\frac{1}{w^2}\right) \\ =\ &\frac{\Gamma\left(1-\frac{q}{2}-s\right)\Gamma\left(\frac{q}{2}+s\right)}{\Gamma\left(2-\frac{q}{2}\right)\Gamma\left(\frac{q}{2}-1\right)}\left(-\frac{1}{w^2}\right)^{2-q-s}\,{}_2F_1\left(s+q-2,\frac{q}{2}-1;\frac{q}{2}+s;w^2\right) \\ &+\frac{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)}{\Gamma(s+1)\Gamma(s+q-2)}\left(-\frac{1}{w^2}\right)^{1-\frac{q}{2}}\,{}_2F_1\left(-s,\frac{q}{2}-1;-s-\frac{q}{2}+2;w^2\right). \end{aligned} \tag{B.3}$$

Using $\hat{h}_1$ from eq. (3.62), $\hat{h}_2$ from eq. (3.63) and $\hat{h}_3$ from eq. (3.64) in sec. (3.4.2), (B.3) can be written as follows:

$$\begin{aligned} &\hat{h}_1-\frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)}\left(-w^2\right)^{-\frac{q}{2}+1}w^{q-2}\hat{h}_3 \\ =\ &-\frac{\Gamma(s+1)\Gamma(s+q-2)\Gamma\left(1-s-\frac{q}{2}\right)}{\Gamma\left(2-\frac{q}{2}\right)\Gamma\left(\frac{q}{2}-1\right)\Gamma\left(s+\frac{q}{2}-1\right)}\left(-w^2\right)^{s+\frac{q}{2}-1}w^{2-q-2s}\hat{h}_2\,. \end{aligned} \tag{B.4}$$

We shall use in (B.4) the following identity,

$$\frac{\Gamma\left(1-s-\frac{q}{2}\right)}{\Gamma\left(2-\frac{q}{2}\right)\Gamma\left(\frac{q}{2}-1\right)}=\frac{\sin\pi\left(\frac{q}{2}-1\right)}{\sin\pi\left(\frac{q}{2}+s\right)}\frac{1}{\Gamma\left(s+\frac{q}{2}\right)}=(-1)^{1+s}\frac{1}{\Gamma\left(s+\frac{q}{2}\right)}\,. \tag{B.5}$$

Even q: In this case, we can use (B.5) in (B.4) to immediately get the required identity in eq. (3.66),

$$\hat{h}_1 = (-1)^{\frac{q}{2}-1} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)} \left(\hat{h}_2 + \hat{h}_3\right) . \tag{B.6}$$

Odd q: In this case, we have to take care of possible branch-cuts while performing manipulations as there is a term $\frac{q}{2}$ appearing as an index. Let us first consider $\mathrm{Im}(w) > 0$ and hence let $w = r\, e^{i\theta}$ with $0 < \theta < \pi$. On the LHS of (B.4), the factors that need care are,

$$\left(-w^2\right)^{-\frac{q}{2}+1} w^{q-2} = r^{-q+2+q-2} \left(e^{2i\theta - i\pi}\right)^{-\frac{q}{2}+1} \left(e^{i\theta}\right)^{q-2} = -e^{i\pi\frac{q}{2}} . \tag{B.7}$$

Note that we accounted for the $-$ sign in $\left(-w^2\right)^{-\frac{q}{2}+1}$ with a factor of $e^{-i\pi}$ and not with $e^{i\pi}$ so that we stay on the same branch (there is a branch cut in $\left(-w^2\right)^{-\frac{q}{2}+1}$ on the negative real axis). Using (B.7), the LHS of (B.4) is now given by,

$$\hat{h}_1 - \left[-e^{i\pi\frac{q}{2}} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)}\right] \hat{h}_3 . \tag{B.8}$$

Similarly, the following factors on the RHS of (B.4) need care,

$$\left(-w^2\right)^{s+\frac{q}{2}-1} w^{2-q-2s} = r^{2s+q-2+2-q-2s} \left(e^{2i\theta - i\pi}\right)^{s+\frac{q}{2}-1} \left(e^{i\theta}\right)^{2-q-2s} = (-1)^{s-1} e^{-i\pi\frac{q}{2}} . \tag{B.9}$$

Using (B.9) and (B.5), the RHS of (B.4) becomes,

$$-e^{-i\pi\frac{q}{2}} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)} \hat{h}_2 . \tag{B.10}$$

Therefore, (B.8) and (B.10) gives us the required identity in eq. (3.74) when $\mathrm{Im}(w) > 0$:

$$\begin{aligned} \hat{h}_1 - c_+ \hat{h}_3 &= -e^{-i\pi\frac{q}{2}} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)} \hat{h}_2 , \\ c_+ &= -e^{i\pi\frac{q}{2}} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)} . \end{aligned} \tag{B.11}$$

When $\mathrm{Im}(w) < 0$, we can proceed exactly as above with the signs of the factors of $e^{\pm i\pi}$ (accounting for the $-$ signs) flipped. This gives us the required identity in eq. (3.74):

$$\begin{aligned} \hat{h}_1 - c_- \hat{h}_3 &= -e^{i\pi\frac{q}{2}} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)} \hat{h}_2 , \\ c_- &= -e^{-i\pi\frac{q}{2}} \frac{\Gamma(s+1)\Gamma(s+q-2)}{\Gamma\left(s+\frac{q}{2}-1\right)\Gamma\left(\frac{q}{2}+s\right)} . \end{aligned} \tag{B.12}$$

Appendix C

The Mellin transform

The Mellin transform of a function $f(x)$ of a real variable is defined by,

$$\mathcal{M}\{f(x)\} \equiv F(z) = \int_0^{\infty} x^{z-1} f(x)\, dx\,, \tag{C.1}$$

where z is a complex number. The Mellin transform is closely related to the Laplace transform and the Fourier transform. A familiar example of a Mellin transform is the transform of e^{-x}, which is the gamma function.

$$\int_0^{\infty} x^{z-1} e^{-x} dx = \Gamma(z)\,. \tag{C.2}$$

$\Gamma(z)$ is meromorphic on the complex plane with simple poles at 0 and all the negative integers.

The inverse Mellin transform is given by,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z)\, x^{-z} dz\,. \tag{C.3}$$

The inverse Mellin transform is unique only upto a choice of the contour specified by the real number c. The inverse Mellin transform of the gamma function with $c > 0$ gives us back e^{-x}.

$$e^{-x} = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \Gamma(z)\, x^{-z} dz\,, \quad \epsilon > 0\,. \tag{C.4}$$

A typical Mellin-Barnes integral over one variable is of the following form,

$$\int_{c-i\infty}^{c+i\infty} \frac{\prod_i \Gamma(z-a_i) \prod_j \Gamma(b_j - z)}{\prod_k \Gamma(z-c_k) \prod_l \Gamma(d_l - z)} x^{-z} dz\,, \tag{C.5}$$

where a_i,b_j,c_k,d_l are complex numbers.

The reciprocal of the gamma function is an entire function and hence all the poles in the integrand of eq. (C.5) are contributed by the numerator. c should be chosen such that the resulting contour separates the poles of $\prod_i \Gamma(t-a_i)$ from the poles of $\prod_j \Gamma(b_j - t)$ and the contour can be deformed near the real axis (if necessary) so as to achieve this separation of poles.

A Mellin representation of the delta function

Let us now discuss a Mellin space version of the delta function which is used in the calculation of Mellin amplitudes for confromal Feynman integrals in sec. 4.7. We want to show the following,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, f(z) \int_0^\infty dt\ t^{z_0-z-1} = \int_{c-i\infty}^{c+i\infty} dz\, f(z)\delta(z-z_o)\,, \quad c = \mathrm{Re}(z_0)\,. \tag{C.6}$$

To prove this, let us first perform a change of variables $t = e^x$ on the left hand side of eq. (C.6) to obtain,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, f(z) \int_{-\infty}^{\infty} dx\ e^{(z_0-z)x}\,. \tag{C.7}$$

$(z_0 - z)$ is purely imaginary along the chosen contour of integration and therefore we can refer to the familiar Fourier space representation of the delta function,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikx}\,, \tag{C.8}$$

to obtain,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz\, f(z) \int_{-\infty}^{\infty} dx\ e^{(z_0-z)x} = \int_{c-i\infty}^{c+i\infty} dz\, f(z)\delta(z_0-z) = f(z_0)\,.$$

This proves the result in eq. (C.6) that in Mellin space, we have the following representation of a delta function,

$$\delta(z-z_o) \equiv \frac{1}{2\pi i}\int_0^\infty dt\ t^{z_0-z-1}\,. \tag{C.9}$$

Appendix D

Tensor structures: fermion four-point function

In sec. 4.2.4, we presented the basis of tensor structures that we use for the four-point function of fermions in eq. (4.23) and eq. (4.24). Now, we shall present some more details regarding this basis of tensor structures. First, let us see how this basis relates to the basis (only parity even basis elements) presented in [164]. The basis in [164] is presented below in eq. (D.1) and eq. (D.2). The elements that are symmetric in crossing $1 \leftrightarrow 3$ are given by:

$$\begin{aligned}
T_1 &= \frac{\langle S_1 S_3\rangle \langle S_2 [X_1, X_3] S_4\rangle}{2X_1\cdot X_3} + \frac{\langle S_2 S_4\rangle \langle S_1 [X_2, X_4] S_3\rangle}{2X_2\cdot X_4}\,, \\
T_2 &= \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_2} - \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_4} - \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_2\cdot X_3} \\
&\quad + \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_3\cdot X_4}\,, \\
T_2 &= \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_2} + \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_4} - \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_2\cdot X_3} \\
&\quad - \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_3\cdot X_4}\,, \\
T_4 &= \frac{\langle S_1 S_3\rangle \langle S_2 [X_1, X_3] S_4\rangle}{2X_1\cdot X_3} - \frac{\langle S_2 S_4\rangle \langle S_1 [X_2, X_4] S_3\rangle}{2X_2\cdot X_4}\,.
\end{aligned} \tag{D.1}$$

The elements that are anti-symmetric in crossing $1 \leftrightarrow 3$ are given by:

$$\begin{aligned}
T_5 &= \langle S_1 S_3\rangle \langle S_2 S_4\rangle\,, \\
T_6 &= \frac{\langle S_1 [X_2, X_4] S_3\rangle \langle S_2 [X_1, X_3] S_4\rangle}{4\,(X_1\cdot X_3)\,(X_2\cdot X_4)}\,, \\
T_7 &= \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_2} + \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_4} + \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_2\cdot X_3} \\
&\quad + \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_3\cdot X_4}\,, \\
T_8 &= \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_2} - \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_1 S_4\rangle}{X_1\cdot X_4} + \frac{\langle S_1 X_2 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_2\cdot X_3} \\
&\quad - \frac{\langle S_1 X_4 S_3\rangle \langle S_2 X_3 S_4\rangle}{X_3\cdot X_4}\,.
\end{aligned} \tag{D.2}$$

With these tensor structures in eq. (D.1) and eq. (D.2), the four-point function is given as,

$$\langle \Psi_1(X_1,S_1)\cdots\Psi_4(X_4,S_4)\rangle = \left(\frac{X_{14}}{X_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}\left(\frac{X_{24}}{X_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}}\sum_i \frac{T_i\tilde{\mathcal{A}}_i(u,v)}{X_{12}^{\frac{\Delta_1+\Delta_2+1}{2}}X_{34}^{\frac{\Delta_3+\Delta_4+1}{2}}}\,. \tag{D.3}$$

Notice from eq. (D.1), eq. (D.2) and eq. (D.3) that the tensor structures from [164] are defined with a different normalization as compared to ours as defined in eq. (4.11). To make the normalization in eq. (D.1) and eq. (D.2) consistent with ours, we shall absorb the factor of $\frac{1}{\sqrt{X_{12}X_{34}}}$ in the tensor structures in eq. (D.1) and eq. (D.2) to obtain $\tilde{T}_i$. p_j^+ eq. (4.23) can then be expressed as linear combinations of $\tilde{T}_i$ with the coefficients being given by conformal invariants.

Let $p_i^+ = \sum_{j=1}^8 A_{ij}\tilde{T}_j$. The matrix A is given as follows,

$$A = \begin{pmatrix}
-\frac{f_3}{4f_1} & \frac{z\bar{z}(f_3-2f_4)}{4f_1} & 0 & 0 & \frac{4z\bar{z}f_4-f_3}{4f_1} & \frac{z+\bar{z}}{4f_1} & \frac{z\bar{z}(f_4-1)}{4f_1} & 0 \\
\frac{\sqrt{z\bar{z}}f_4}{2f_1} & \frac{\sqrt{z\bar{z}}f_5}{8f_1} & \frac{\sqrt{z\bar{z}}}{8} & 0 & -\frac{\sqrt{z\bar{z}}(2f_1+f_8)}{2f_1} & -\frac{\sqrt{z\bar{z}}}{2f_1} & -\frac{\sqrt{z\bar{z}}(f_4^2-1)}{8f_1} & \frac{\sqrt{z\bar{z}}}{8} \\
\frac{\sqrt{z\bar{z}}f_4(f_4+3)}{4f_1\sqrt{f_2}} & \frac{\sqrt{z\bar{z}}f_6}{4f_1\sqrt{f_2}} & 0 & \frac{\sqrt{z\bar{z}}}{4\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}f_9}{4f_1\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}(f_4+3)}{4f_1\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}f_{11}}{4f_1\sqrt{f_2}} & 0 \\
-\frac{f_3+8f_4}{4f_1} & \frac{f_7}{4f_1} & 0 & 0 & \frac{f_{10}}{4f_1} & \frac{f_4+9}{4f_1} & \frac{f_{12}}{4f_1} & 0 \\
\frac{\sqrt{z\bar{z}}f_4(f_4-1)}{4f_1\sqrt{f_2}} & -\frac{\sqrt{f_2}\sqrt{z\bar{z}}f_3}{4f_1} & 0 & \frac{\sqrt{z\bar{z}}}{4\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}(f_4-1)f_8}{4f_1\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}(f_4-1)}{4f_1\sqrt{f_2}} & -\frac{\sqrt{f_2}\sqrt{z\bar{z}}(z+\bar{z})}{4f_1} & 0 \\
-\frac{\sqrt{z\bar{z}}f_4}{2f_1} & \frac{\sqrt{z\bar{z}}(f_4-1)f_3}{8f_1} & \frac{\sqrt{z\bar{z}}}{8} & 0 & \frac{\sqrt{z\bar{z}}f_8}{2f_1} & \frac{\sqrt{z\bar{z}}}{2f_1} & \frac{\sqrt{z\bar{z}}(f_4^2-1)}{8f_1} & -\frac{\sqrt{z\bar{z}}}{8} \\
-\frac{\sqrt{z\bar{z}}f_4}{2f_1} & \frac{\sqrt{z\bar{z}}(f_4-1)f_3}{8f_1} & -\frac{\sqrt{z\bar{z}}}{8} & 0 & \frac{\sqrt{z\bar{z}}f_8}{2f_1} & \frac{\sqrt{z\bar{z}}}{2f_1} & \frac{\sqrt{z\bar{z}}(f_4^2-1)}{8f_1} & \frac{\sqrt{z\bar{z}}}{8} \\
\frac{\sqrt{z\bar{z}}f_4(f_4-1)}{4f_1\sqrt{f_2}} & -\frac{\sqrt{f_2}\sqrt{z\bar{z}}f_3}{4f_1} & 0 & -\frac{\sqrt{z\bar{z}}}{4\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}(f_4-1)f_8}{4f_1\sqrt{f_2}} & -\frac{\sqrt{z\bar{z}}(f_4-1)}{4f_1\sqrt{f_2}} & -\frac{\sqrt{f_2}\sqrt{z\bar{z}}(z+\bar{z})}{4f_1} & 0
\end{pmatrix}, \tag{D.4}$$

with

$$\begin{aligned}
f_1 &= (z-\bar{z})^2\,, & f_2 &= (z-1)(\bar{z}-1)\,,\\
f_3 &= 4z\bar{z}-z-\bar{z}\,, & f_4 &= z+\bar{z}-1\,,\\
f_5 &= 3f_1-4z\bar{z}f_4+2f_3\,, & f_6 &= f_1-f_3(z\bar{z}-1)\,,\\
f_7 &= (z\bar{z}-4)f_3+6z\bar{z}f_4-4f_1\,, & f_8 &= (2z-1)(2\bar{z}-1)\,,\\
f_9 &= (4z\bar{z}-1)f_4+6f_1+f_8\,, & f_{10} &= (4z\bar{z}-1)f_4+32f_1+7f_8\,,\\
f_{11} &= z\bar{z}f_4+f_1+f_2-1\,, & f_{12} &= z\bar{z}\,(f_4-1)+4f_1+2f_8-2\,,
\end{aligned} \tag{D.5}$$

where z and $\bar{z}$ are the familiar Dolan-Osborn coordinates [173].

From the matrix in eq. (D.4), we automatically get the change of basis from $\mathcal{A}_i$ (components of the correlator in our basis) to $\tilde{\mathcal{A}}_i$ (components of the correlator in [164]):

$$\tilde{\mathcal{A}}_i(u,v) = A^t_{ij}\mathcal{A}_j(u,v)\,, \tag{D.6}$$

where A^t is the transpose of the matrix A. Note that the factor of $\frac{1}{f_1}$ in many of the elements in the matrix in eq. (D.4) gives the spurious singularity that makes T_i (or $\tilde{T}_i$) unsuitable as a basis for defining the Mellin amplitude.

Let us now see how crossing acts on the basis we have chosen in eq. (4.23) and eq. (4.24). The change of basis is given by a 16×16 matrix (say R) but since parity even and parity odd structures do not transform into each other, this matrix has a block diagonal form with two 8×8 blocks (say R_e and R_o for the even and odd parts respectively). Let us first look at the block dealing with the parity even structures. Let $p_i^+|_{1\leftrightarrow 3} = \sum_{j=1}^{8} R_{ij} p_j^+ \,, i \in \{1, \cdots, 8\}$. The corresponding block of the matrix R is as follows:

$$R_e = \begin{pmatrix} \frac{\sqrt{z\bar{z}}}{\sqrt{f_2}} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{z\bar{z}}}{4} & 0 & \frac{1}{2}\sqrt{f_2} & \frac{\sqrt{z\bar{z}}}{4} & -\frac{1}{2}\sqrt{f_2} & 1 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{\sqrt{f_2}}{\sqrt{z\bar{z}}} & 0 & 0 & 0 \\ \frac{2\sqrt{z\bar{z}}}{\sqrt{f_2}} & 0 & \frac{1}{2} & -\frac{\sqrt{z\bar{z}}}{\sqrt{f_2}} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{\sqrt{f_2}}{\sqrt{z\bar{z}}} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{z\bar{z}}}{4} & 1 & \frac{1}{2}\sqrt{f_2} & \frac{\sqrt{z\bar{z}}}{4} & -\frac{1}{2}\sqrt{f_2} & 0 & 0 & 0 \\ \frac{f_2-1}{4\sqrt{z\bar{z}}} & -1 & -\frac{(f_2-f_4-2)\sqrt{f_2}}{2z\bar{z}} & \frac{-f_2+1}{4\sqrt{z\bar{z}}} & \frac{(f_2-f_4)\sqrt{f_2}}{2z\bar{z}} & -\frac{f_2}{z\bar{z}} & -\frac{f_2}{z\bar{z}} & \frac{\sqrt{f_2}}{z\bar{z}} \\ -\frac{1}{4} & 0 & \frac{\sqrt{f_2}}{\sqrt{z\bar{z}}} & \frac{1}{4} & -\frac{\sqrt{f_2}}{\sqrt{z\bar{z}}} & 0 & 0 & \frac{\sqrt{f_2}}{\sqrt{z\bar{z}}} \end{pmatrix}, \tag{D.7}$$

with f_i as given in (D.5).

We can already perform a small consistency check of our results eq. (D.4) and eq. (D.7). As mentioned before, the basis elements in eq. (D.1) and eq. (D.2) are symmetric and anti-symmetric respectively under crossing $1 \leftrightarrow 3$. From this we can obtain,

$$\tilde{T}_i\big|_{1\leftrightarrow 3} = \sqrt{\frac{u}{v}} \sum_{j=1}^{8} \tilde{I}_{ij} \tilde{T}_j = \sqrt{\frac{u}{v}} \sum_{j=1}^{8} \tilde{I}_{ij} \sum_{k=1}^{8} A^{-1}_{jk} p_k^+ \,, \tag{D.8}$$

where the matrix I is diagonal with elements $\{1, 1, 1, 1, -1, -1, -1, -1\}$.

But we can also write, for $i \in \{1, \cdots, 8\}$

$$\tilde{T}_i\big|_{1\leftrightarrow 3} = \sum_{j=1}^{8} \left(A^{-1}_{ij} p_j^+ \right) \bigg|_{1\leftrightarrow 3} = \sum_{j=1}^{8} \left(A^{-1}_{ij}\big|_{1\leftrightarrow 3} \right) \sum_{k=1}^{8} R_{jk} p_j^+ \,. \tag{D.9}$$

From eq. (D.8) and eq. (D.9), we see that the following must be satisfied $\forall i, k \in \{1, \cdots, 8\}$:

$$\sqrt{\frac{u}{v}} \sum_{j=1}^{8} \tilde{I}_{ij} A^{-1}_{jk} = \sum_{j=1}^{8} \left(A^{-1}_{ij}\big|_{1\leftrightarrow 3} \right) R_{jk} \,. \tag{D.10}$$

Using eq. (D.4) and eq. (D.7), we can indeed check that eq. (D.10) is true.

Finally let us see how crossing acts on the parity odd elements eq. (4.24) of our basis. Let $p_i^-|_{1\leftrightarrow 3} = \sum_{j=9}^{16} R_{ij} p_j^- \,, i \in \{9, \cdots, 16\}$. The corresponding block of the matrix R is as follows:

$$R_o = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}. \qquad \text{(D.11)}$$

Appendix E

Mellin amplitude from the reduced Mellin amplitude

We present here the relations between the reduced Mellin amplitude $\{\bar{M}_i\}$ and the Mellin amplitude $\{\mathcal{M}_i\}$.

E.1 Fermion-scalar four-point function

$$
\begin{aligned}
\mathcal{M}_1 &= \bar{M}_1 \left[\Gamma(s_{12}+1)\Gamma(s_{13})\Gamma(s_{14})\Gamma(s_{23})\Gamma(s_{24})\Gamma(s_{34})\right]^{-1}, \\
\mathcal{M}_2 &= \bar{M}_2 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma(s_{14})\Gamma(s_{23})\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1}, \\
\mathcal{M}_3 &= \bar{M}_3 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma(s_{14})\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma(s_{24})\Gamma(s_{34})\right]^{-1}, \\
\mathcal{M}_4 &= \bar{M}_4 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma(s_{13})\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma(s_{23})\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma(s_{34})\right]^{-1}.
\end{aligned} \tag{E.1}
$$

E.2 Fermion four-point function

$$
\begin{aligned}
\mathcal{M}_1 &= \bar{M}_1 \left[\Gamma(s_{12}+1)\Gamma(s_{13})\Gamma(s_{14})\Gamma(s_{23})\Gamma(s_{24})\Gamma(s_{34}+1)\right]^{-1}, \\
\mathcal{M}_2 &= \bar{M}_2 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma(s_{14})\Gamma(s_{23})\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1}, \\
\mathcal{M}_3 &= \bar{M}_3 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma(s_{13})\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma(s_{24})\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1}, \\
\mathcal{M}_4 &= \bar{M}_4 \left[\Gamma(s_{12}+1)\Gamma(s_{13})\Gamma(s_{14})\Gamma(s_{23})\Gamma(s_{24})\Gamma(s_{34}+1)\right]^{-1},
\end{aligned}
$$

$$\mathcal{M}_5 = \bar{M}_5 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_6 = \bar{M}_6 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_7 = \bar{M}_7 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_8 = \bar{M}_8 \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_9 = \bar{M}_9 \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_{10} = \bar{M}_{10} \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_{11} = \bar{M}_{11} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$

$$\mathcal{M}_{12} = \bar{M}_{12} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$

$$\mathcal{M}_{13} = \bar{M}_{13} \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_{14} = \bar{M}_{14} \left[\Gamma\left(s_{12}+1\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+\frac{1}{2}\right)\right]^{-1},$$

$$\mathcal{M}_{15} = \bar{M}_{15} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}+\frac{1}{2}\right)\Gamma\left(s_{14}\right)\Gamma\left(s_{23}+\frac{1}{2}\right)\Gamma\left(s_{24}\right)\Gamma\left(s_{34}+1\right)\right]^{-1},$$

$$\mathcal{M}_{16} = \bar{M}_{16} \left[\Gamma\left(s_{12}+\frac{1}{2}\right)\Gamma\left(s_{13}\right)\Gamma\left(s_{14}+\frac{1}{2}\right)\Gamma\left(s_{23}\right)\Gamma\left(s_{24}+\frac{1}{2}\right)\Gamma\left(s_{34}+1\right)\right]^{-1}.$$

Appendix F

Further results on the pole structure of Mellin amplitudes

We provide some more results on the pole structure of Mellin amplitudes here.

F.1 u-channel poles in fermion-scalar four-point Mellin amplitude

The u-channel poles in the fermion-scalar four-point function are summarised in table. F.1.

Component of M.A.	**Location of Poles**	**Residues** $\sim$
$\mathcal{M}_1$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^{+}_{\psi_1\phi_4\Psi_l}\lambda^{+}_{\Psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^{-}_{\psi_1\phi_4\Psi_l}\lambda^{-}_{\Psi_l\phi_3\psi_2}$
$\mathcal{M}_2$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^{+}_{\psi_1\phi_4\Psi_l}\lambda^{+}_{\Psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^{-}_{\psi_1\phi_4\Psi_l}\lambda^{-}_{\Psi_l\phi_3\psi_2}$
$\mathcal{M}_3$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^{+}_{\psi_1\phi_4\Psi_l}\lambda^{-}_{\Psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^{-}_{\psi_1\phi_4\Psi_l}\lambda^{+}_{\Psi_l\phi_3\psi_2}$
$\mathcal{M}_4$	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^{+}_{\psi_1\phi_4\Psi_l}\lambda^{-}_{\Psi_l\phi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^{-}_{\psi_1\phi_4\Psi_l}\lambda^{+}_{\Psi_l\phi_3\psi_2}$

Table F.1: Fermion-scalar four-point function: u-channel poles.

F.2 Crossed channel poles in the fermion four-point Mellin amplitude

Corresponding to each integer spin l primary $\mathcal{O}_l$ of twist τ contributing to the $\psi_1\psi_3$ and $\psi_2\psi_4$ OPE, the Mellin amplitude has t-channel poles and residues as summarised in table. F.2.

Component of M.A.	**Location of Poles**	**Residues ∼**
$\mathcal{M}_1$, $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_5$	$t = \tau - 1 + 2k$	$\lambda^{1}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{1}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{1}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{1}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$
	$t = \tau + 2k$	$\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_2$, $\mathcal{M}_6$, $\mathcal{M}_7$	$t = \tau + 2k$	$\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{1}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$
	$t = \tau + 1 + 2k$	$\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_8$	$t = \tau + 1 + 2k$	$\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{1}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$
	$t = \tau + 2 + 2k$	$\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_9$, $\mathcal{M}_{11}$, $\mathcal{M}_{13}$, $\mathcal{M}_{15}$	$t = \tau + 2k$	$\lambda^{1}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{1}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$
	$t = \tau + 1 + 2k$	$\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$
$\mathcal{M}_{10}$, $\mathcal{M}_{12}$, $\mathcal{M}_{14}$, $\mathcal{M}_{16}$	$t = \tau + 2k$	$\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{1}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{1}_{\mathcal{O}_l\psi_2\psi_4}$ $\lambda^{3}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{4}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{2}_{\mathcal{O}_l\psi_2\psi_4}$
	$t = \tau + 1 + 2k$	$\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{3}_{\mathcal{O}_l\psi_2\psi_4}$, $\lambda^{2}_{\psi_1\psi_3\mathcal{O}_l}\lambda^{4}_{\mathcal{O}_l\psi_2\psi_4}$

Table F.2: Fermion four-point function: t-channel poles.

The u-channel poles are summarised in table. F.3. When the exchanged operator is a scalar $l = 0$, we should take all structure constants apart from λ^1, λ^3 to be zero.

Component of M.A.	**Location of Poles**	**Residues** $\sim$
$\mathcal{M}_1$	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_2$, $\mathcal{M}_4$	$s+t=\sum_i \tau_i - \tau + 1 - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_3$, $\mathcal{M}_5$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_6$, $\mathcal{M}_7$	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 2 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_8$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^3_{\psi_1\psi_3\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_9$, $\mathcal{M}_{12}$, $\mathcal{M}_{13}$, $\mathcal{M}_{16}$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^1_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
$\mathcal{M}_{10}$, $\mathcal{M}_{11}$, $\mathcal{M}_{14}$, $\mathcal{M}_{15}$	$s+t=\sum_i \tau_i - \tau - 2k$	$\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^1_{\mathcal{O}_l\psi_3\psi_2}$ $\lambda^3_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^4_{\psi_1\psi_4\mathcal{O}_l}\lambda^2_{\mathcal{O}_l\psi_3\psi_2}$
	$s+t=\sum_i \tau_i - \tau - 1 - 2k$	$\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^3_{\mathcal{O}_l\psi_3\psi_2}$, $\lambda^2_{\psi_1\psi_4\mathcal{O}_l}\lambda^4_{\mathcal{O}_l\psi_3\psi_2}$

Table F.3: Fermion four-point function: u-channel poles.

Appendix G

Mellin amplitudes for conformal integrals: a recursive method

The calculation of Mellin amplitudes associated to conformal integrals presented in [99] involved the following steps: Each propagator (internal and external) in a given diagram was expressed in a Schwinger parametrized manner, and then the position space integrals over the interaction vertices were evaluated successively. Following this, we would be left with a Schwinger parameter integral that could be simplified drastically using the conformality of the overall integral, and the resulting integral could be evaluated exactly to give the Mellin amplitude as a product of beta functions.

In the present case, when the position space conformal integral has fermionic legs, the simplifications in the Schwinger parameter integral using the conformality condition are not as good, consequently the final Schwinger parameter integrals are complicated. Hence we shall apply a recursive method which allows us to reduce the calculation of any Feynman diagram to the calculation of a series of contact interaction diagrams. This technique was developed by Arnab Rudra for scalar conformal integrals while working on [99].

To illustrate the procedure, we apply the recursive method to a simple example: a four-point diagram of scalars with a scalar propagator as in fig. G.1. The conformal integral

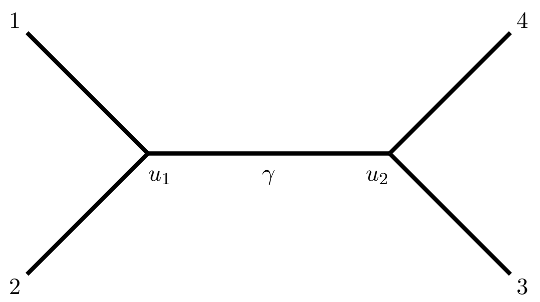


Figure G.1: Four external scalars: Scalar exchange corresponding to this diagram is given by,

$$I^{\phi_1\phi_2}_{\phi_3\phi_4} = \int \mathcal{D}u_1 \int \mathcal{D}u_2 \prod_{i=1}^{2} \frac{\Gamma(\Delta_i)}{|x_i - u_1|^{2\Delta_i}} \prod_{i=3}^{4} \frac{\Gamma(\Delta_i)}{|x_i - u_2|^{2\Delta_i}} \frac{1}{|u_1 - u_2|^{2\gamma}}. \tag{G.1}$$

The conformality condition is $\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4 = d - \gamma$. Now we shall treat the second interaction vertex u_2 like it existed indepedently as a contact interaction diagram with the "external" legs at x_1, x_2 and u_1. This is depicted pictorially in fig. G.2.

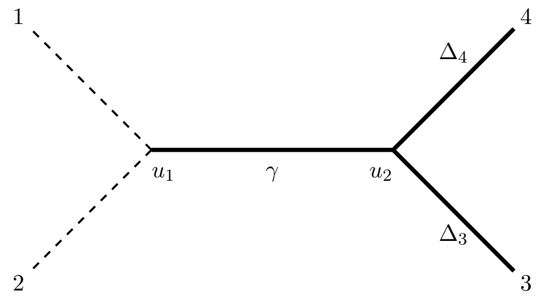


Figure G.2: Recursive method: First step

We know the Mellin-Barnes representation of the contact interaction conformal integral with scalar legs [97, 99, 220].

$$\int \mathcal{D}u_2 \prod_{i=3}^{4} \frac{\Gamma(\Delta_i)}{|x_i - u_2|^{2\Delta_i}} \frac{1}{|u_1 - u_2|^{2\gamma}} = \left(\prod_{i=3}^{4} \int_{c_{iu}-i\infty}^{c_{iu}+i\infty} (ds_{iu}) \right) \frac{1}{\Gamma(\gamma)} \int_{\bar{c}_{34}-i\infty}^{\bar{c}_{34}+i\infty} (d\bar{s}_{34})$$
$$\frac{\Gamma(s_{iu})}{|x_i - u_1|^{2s_{iu}}} \frac{\Gamma(\bar{s}_{34})}{|x_{34}|^{2\bar{s}_{34}}} \prod_{i=3}^{4} \hat{\delta}(\Delta_i - \bar{s}_{34} - s_{iu}) \, \hat{\delta}(\gamma - s_{3u} - s_{4u}). \tag{G.2}$$

The contours of the Mellin-Barnes integrals are such that the series poles of the gamma functions are not separated and that the integrals converge (see [99]). Now we can plug the result eq. (G.2) back in eq. (G.1) to obtain the second contact interaction conformal integral that we need to evaluate. The legs are now given by (x_1, u_1) with dimension Δ_1, (x_2, u_1) with dimension Δ_2, (x_3, u_1) with "dimension" s_{3u} and (x_4, u_4) with "dimension" s_{4u}. This is represented pictorially in fig. G.3. Using the $2\pi i \; \delta(\gamma - s_{3u} - s_{4u}) = \hat{\delta}(\gamma - s_{3u} - s_{4u})$ in eq. (G.2), we also get the required conformality condition for this integral $\Delta_1 + \Delta_2 + s_{3u} + s_{4u} = d$.

Once again, we use the known result for the contact interaction of scalars, and plug it

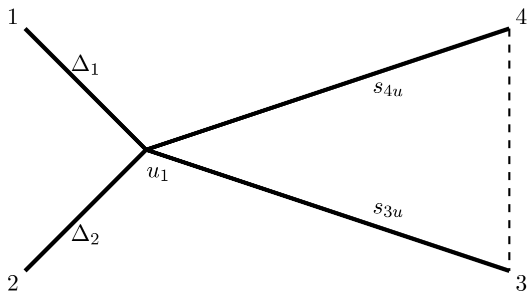


Figure G.3: Recursive method: Second step

back into eq. (G.2), to obtain,

$$\begin{aligned}
I^{\phi_1\phi_2}_{\phi_3\phi_4} &= \prod_{1\le i<l}^{4} \int_{\tilde{c}_{il}-i\infty}^{\tilde{c}_{il}+i\infty} (d\tilde{s}_{il}) \frac{\Gamma(\tilde{s}_{il})}{|x_{il}|^{2\tilde{s}_{il}}} \int_{\bar{c}_{34}-i\infty}^{\bar{c}_{34}+i\infty} (d\bar{s}_{34}) \frac{\Gamma(\bar{s}_{34})}{|x_{34}|^{2\bar{s}_{34}}} \frac{1}{\Gamma(\gamma)} \\
&\left(\prod_{i=3}^{4} \int_{c_{iu}-i\infty}^{c_{iu}+i\infty} (ds_{iu})\right) \hat{\delta}\left(s_{3u} - \tilde{s}_{13} - \tilde{s}_{23} - \tilde{s_{34}}\right) \hat{\delta}\left(s_{4u} - \tilde{s}_{14} - \tilde{s}_{24} - \tilde{s_{34}}\right) \\
&\hat{\delta}\left(\Delta_1 - \tilde{s}_{12} - \tilde{s}_{13} - \tilde{s}_{14}\right) \hat{\delta}\left(\Delta_2 - \tilde{s}_{12} - \tilde{s}_{23} - \tilde{s}_{24}\right) \\
&\prod_{i=3}^{4} \hat{\delta}\left(\Delta_i - \bar{s}_{34} - s_{iu}\right) \hat{\delta}\left(\gamma - s_{3u} - s_{4u}\right).
\end{aligned}$$

The Mellin variables introduced in the second step are indicated with the tilde. Next, we integrate out the s_{iu} using the delta functions, rename $\tilde{s}_{ij} = s_{ij}$ for $(i,j) \neq (3,4)$ and take $\bar{s}_{34} = s_{34} - \tilde{s}_{34}$, such that we obtain,

$$\begin{aligned}
I^{\phi_1\phi_2}_{\phi_3\phi_4} &= \prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il}) \frac{\Gamma(s_{il})}{|x_{il}|^{2s_{il}}} \prod_{i=1}^{4} \hat{\delta}\left(\Delta_i - \sum_{k=1,k\neq i}^{4} s_{ik}\right) \\
&\frac{1}{\Gamma(\gamma)} \int_{\tilde{c}_{34}-i\infty}^{\tilde{c}_{34}+i\infty} (d\tilde{s}_{34}) \frac{\Gamma(\tilde{s}_{34})\Gamma(s_{34}-\tilde{s}_{34})}{\Gamma(s_{34})} \hat{\delta}\left(\gamma - K_{12,34} - 2\tilde{s}_{34}\right). \qquad (G.3)
\end{aligned}$$

We have introduced the notation $K_{ij,kl} = s_{ik} + s_{il} + s_{jk} + s_{jl}$. Now we can integrate over $\tilde{s_{34}}$ and simplify the result to obtain,

$$\begin{aligned}
I^{\phi_1\phi_2}_{\phi_3\phi_4} &= \prod_{1\le i<l}^{4} \int_{c_{il}-i\infty}^{c_{il}+i\infty} (ds_{il}) \frac{\Gamma(s_{il})}{|x_{il}|^{2s_{il}}} \prod_{i=1}^{4} \hat{\delta}\left(\Delta_i - \sum_{k=1,k\neq i}^{4} s_{ik}\right) \\
&\frac{1}{2\Gamma(\gamma)} B\left(\frac{\gamma - K_{12,34}}{2}, \frac{d-2\gamma}{2}\right). \qquad (G.4)
\end{aligned}$$

We simplified the second argument of the beta function using the conformality condition and the constraints imposed by the delta functions:

$$d = \gamma + \Delta_3 + \Delta_4 \qquad \text{and} \qquad \Delta_i = \sum_{k=i,i\neq k}^{4} s_{ik}.$$

Eq. (G.4) is the familiar result obtained for the scalar propagator in Mellin space as obtained in [97, 99].

In general for more complicated Feynman diagrams, one can carry on this procedure and use the result for the contact interaction at each step. This would typically give a nested Mellin-Barnes integral over beta functions. All of the technicalities in the method presented in [99], for example making a suitable choice for the order of integration over the vertices, still continue to hold. Let us summarize the differences between the two methods: we are trading some nested Schwinger parameter integrals for some nested Mellin-Barnes integrals. Thus, in the case of scalars, this technique does not offer any simplifications over the method presented in [99]. However for conformal integrals with legs with spin, the Schwinger parameter integrals are particularly difficult and therefore this method is helpful. One has to do a set of Schwinger parameter integrals while calculating the Mellin amplitude associated with the contact interaction diagram, but for all other Feynman diagrams there are no further Schwinger parameter integrals to be evaluated.

Appendix H

OPE coefficients and three-point function coefficients on the worldsheet

In the context of the calculations presented in chap. 5, we want to obtain the relationship between the three-point function coefficients $\bar{F}_{123}^{(0,0,l)}$ and the OPE coefficients $F_{123}^{(0,0,l)}$ in the worldsheet theory. Let us consider the three-point function of one spin l and two spin zero vertex operators and approximate it with the OPE of the vertex operators $\mathcal{V}_1(x_1)$ and $\mathcal{V}_2(x_2)$ given by eq. (5.38). This gives us,

$$\begin{aligned}
&\langle \mathcal{V}_1(x_1, z_1)\mathcal{V}_2(0,0)\mathcal{V}_{\Delta_3,q_3}^{\mu_1\cdots\mu_l}(x_3, z_3)\rangle \\
&\approx \sum_q \int_C d\Delta \frac{|z_1|^{-(h(1)+h(2)-h(\Delta,q))}}{|x_1|^{\alpha+\beta+\gamma-D+l}} F_{12\Delta}^{(0,0,l)} \langle \mathcal{V}_{\Delta,q}^{\nu_1\cdots\nu_l}(0,0)\mathcal{V}_{\Delta_3,q_3}^{\mu_1\cdots\mu_l}(x_3, z_3)\rangle G_{\nu_1\cdots\nu_l}(x_1)\,,
\end{aligned} \tag{H.1}$$

where $G_{\nu_1\cdots\nu_l}$ is given by eq. (5.39) and α, β, γ are given by eq. (5.23).

Conformal symmetry on the boundary and on the worldsheet fixes the two-point function in eq. (H.1) to be of the following form:

$$\left\langle \mathcal{V}_1^{\mu_1\cdots\mu_l}(x_1, z_1)\mathcal{V}_{2,\nu_1\cdots\nu_l}(x_2, z_2)\right\rangle = \frac{\delta_{q_1,q_2}}{|z_{12}|^{2h_1}} \left[D_l(\Delta_1)\delta(\Delta_1-\Delta_2)\frac{J_{\nu_1}^{(\mu_1}\cdots J_{\nu_l}^{\mu_l)} - \text{Traces}}{|x_{12}|^{2\Delta_1}}\right]\,, \tag{H.2}$$

$D_l(\Delta)$ being a constant for a given operator. q_1 and q_2 are additional discrete quantum numbers associated with the vertex operators (suppressed on the left hand side). Therefore the three-point function in eq. (H.1) can be approximated as,

$$\begin{aligned}
&\langle \mathcal{V}_1(x_1, z_1)\mathcal{V}_2(0,0)\mathcal{V}_{\Delta_3,q_3}^{\mu_1\cdots\mu_l}(x_3; z_3)\rangle \\
&= F_{123}^{(0,0,l)} D_l(\Delta_3)\frac{|z_1|^{-(h(1)+h(2)-h(\Delta_3,q_3))}}{|x_1|^{\alpha+\beta+\gamma-D+l}} \frac{\left(J_{(\nu_1}^{\mu_1}(x_3)\cdots J_{\nu_l)}^{\mu_l}(x_3) - \text{Traces}\right)}{|z_3|^{2h_3}\ |x_3|^{2\Delta_3}} G^{\nu_1\cdots\nu_l}(x_1)\,.
\end{aligned} \tag{H.3}$$

However conformal symmetry fixes the form of this three-point function directly as follows,

$$\begin{aligned}
&\langle \mathcal{V}_1(x_1; z_1)\mathcal{V}_4(0;0)\mathcal{V}^{\mu_1\cdots\mu_l}_{\Delta_3,q_3}(x_3;z_3)\rangle \\
= \quad & \frac{\bar{F}^{(0,0,l)}_{123}}{|z_1|^{h_1+h_1-h_3}|z_3|^{2h_3}} \frac{\Big(V^{\mu_1}(x_1,0,x_3)\cdots V^{\mu_l}(x_1,0,x_3) - \text{Traces}\Big)}{|x_1|^{\Delta_1+\Delta_2-\Delta_3+l}|x_3|^{2(\Delta_3-l)}} \,.
\end{aligned} \tag{H.4}$$

Simplifying the tensor structure in eq. (H.4) and comparing with eq. (H.3) , we obtain the desired result:

$$\bar{F}^{(0,0,l)}_{123} = \left[\pi^{d/2}\Lambda_l(\beta,\gamma)D_l(\Delta_3)\frac{\Gamma\left(\frac{d-\beta}{2}\right)\Gamma\left(\frac{d-\gamma}{2}\right)\Gamma\left(\frac{\beta+\gamma-d}{2}+l\right)}{\Gamma\left(\frac{\beta}{2}+l\right)\Gamma\left(\frac{\gamma}{2}+l\right)\Gamma\left(\frac{2d-\beta-\gamma}{2}\right)}\right] F^{(0,0,l)}_{123} \,, \tag{H.5}$$

where $\Lambda_l(\beta,\gamma)$ is defined in equation (5.35).

Appendix I

Normalization of vertex operators in the worldsheet CFT

It is important to fix the relative normalization between dual worldsheet vertex operators and boundary CFT operators in order that the relations between the corresponding OPE coefficients in the worldsheet and boundary theories presented in chap. 5 are meaningful. We have fixed the normalization of the boundary CFT two-point functions of spin l operators to be $K_l(\Delta)$ with $K_0(\Delta) = 1$, and this fixes the relative normalization of the vertex operators automatically.

The two-point function of vertex operators (carrying Lorentz indices of the boundary spacetime as global symmetry labels) is given by eq. (H.2) upto a proportionality constant that is to be fixed. We can set the worldsheet coordinates at $z = 0$ and $z = 1$ in eq. (H.2) and compute the boundary CFT two-point function as follows,

$$\begin{aligned} \left\langle \mathcal{O}_1^{\mu_1\cdots\mu_l}(x_1)\mathcal{O}_{2\nu_1\cdots\nu_l}(x_2)\right\rangle &= \frac{1}{V_{\text{conf}}}\left\langle \mathcal{V}_1^{\mu_1\cdots\mu_l}(x_1, z_1 = 1)\mathcal{V}_{2\nu_1\cdots\nu_l}(x_2, z_2 = 0)\right\rangle , \\ &= f(\Delta)\left[\frac{D_l(\Delta)\big(J_{\nu_1}^{(\mu_1}\cdots J_{\nu_l}^{\mu_l)} - \text{Traces}\big)}{|x_{12}|^{2\Delta}}\right] , \end{aligned} \tag{I.1}$$

where J^μ_ν is given by eq. (5.13).

We have divided the right hand side by the volume of the conformal group V_{conf} on the sphere as we have fixed the worldsheet coordinates. This factor cancels the divergence from the delta function in eq. (H.2) upto a factor $f(\Delta)$. This factor can be explicitly determined when we know the spectrum of the theory as in the case of $\text{AdS}_3/\text{CFT}_2$ correspondence (see, e.g., [123]).

Matching eq. (I.1) with eq. (5.41) fixes the relative normalization between the boundary and the worldsheet vertex operators as follows,

$$\mathcal{O}_\Delta(x) = \frac{1}{a_\Delta^{(l)}}\int d^2z\ \mathcal{V}_\Delta(x, z)\,, \tag{I.2}$$

where,

$$a_\Delta^{(l)} \equiv \sqrt{\frac{f(\Delta)D_l(\Delta)}{K_l(\Delta)}}\,. \tag{I.3}$$

Appendix J

Useful integrals

We provide here some integrals (the last two of which are adopted from [173]) that have been useful in chap. 5.

- $$I(a,b) \equiv \int \frac{d^d y}{|y|^a \, |y-\hat{x}|^b} = \frac{\pi^{d/2}\Gamma\left(\frac{d-a}{2}\right)\Gamma\left(\frac{d-b}{2}\right)\Gamma\left(\frac{a+b-d}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{2d-a-b}{2}\right)} \,. \tag{J.1}$$

  Note that this is manifestly symmetric in a and b.

- $$\begin{aligned} &\int \frac{d^d y}{|y_1-y|^{a_1}|y_2-y|^{a_2}|y_3-y|^{a_3}} \\ &\quad = \frac{\pi^{d/2}}{|y_{12}|^{d-a_3}|y_{23}|^{d-a_1}|y_{13}|^{d-a_2}} \frac{\Gamma\left(\frac{d-a_1}{2}\right)\Gamma\left(\frac{d-a_2}{2}\right)\Gamma\left(\frac{d-a_3}{2}\right)}{\Gamma\left(\frac{a_1}{2}\right)\Gamma\left(\frac{a_2}{2}\right)\Gamma\left(\frac{a_3}{2}\right)} \,. \end{aligned} \tag{J.2}$$

- $$\begin{aligned} &\int d^d y \, \frac{J^{\mu_1}_{(\nu_1}(x-y)\cdots J^{\mu_\ell}_{\nu_\ell)}(x-y) V^{\nu_1}(x_1,x_2,y)\cdots V^{\nu_\ell}(x_1,x_2,y)}{|x_1-y|^a|x_2-y|^b|x-y|^c} \\ &\quad = \pi^{d/2}\frac{\Lambda_\ell(a,b)\Gamma\left(\frac{d-a}{2}\right)\Gamma\left(\frac{d-b}{2}\right)\Gamma\left(\frac{d-c}{2}\right)}{\Gamma\left(\frac{a}{2}+\ell\right)\Gamma\left(\frac{b}{2}+\ell\right)\Gamma\left(\frac{c}{2}+\ell\right)} \frac{V^{\mu_1}(x_1,x_2,x)\cdots V^{\mu_\ell}(x_1,x_2,x)}{|x-x_1|^{a+c-d}|x-x_2|^{b+c-d}|x_1-x_2|^{d-c}} \,, \end{aligned} \tag{J.3}$$

  where $a+b+c=2d-2\ell$, J^μ_ν is given in eq. (5.13), V^ν is given in eq. (5.32) and $\Lambda_\ell(a,b)$ is given in eq. (5.35).

- $$\begin{aligned} &\int d^d y \, \frac{V_{\mu_1}(x_1,x_2,y)\cdots V_{\mu_\ell}(x_1,x_2,y)}{|x_1-y|^a|x_2-y|^b} \\ &= \pi^{d/2}\Lambda_\ell(a,b)\frac{\Gamma\left(\frac{a+b-d}{2}+\ell\right)\Gamma\left(\frac{d-a}{2}\right)\Gamma\left(\frac{d-b}{2}\right)}{\Gamma\left(d-\frac{a+b}{2}\right)\Gamma\left(\frac{a}{2}+\ell\right)\Gamma\left(\frac{b}{2}+\ell\right)} \frac{(x_{12})_{\mu_1}\cdots(x_{12})_{\mu_\ell}}{|x_{12}|^{2\left(\frac{a+b-d}{2}+\ell\right)}} \,, \end{aligned} \tag{J.4}$$

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